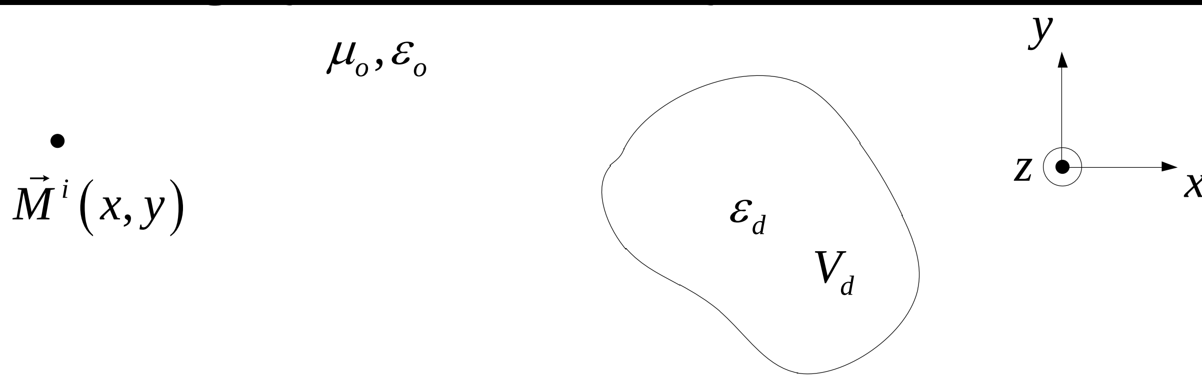
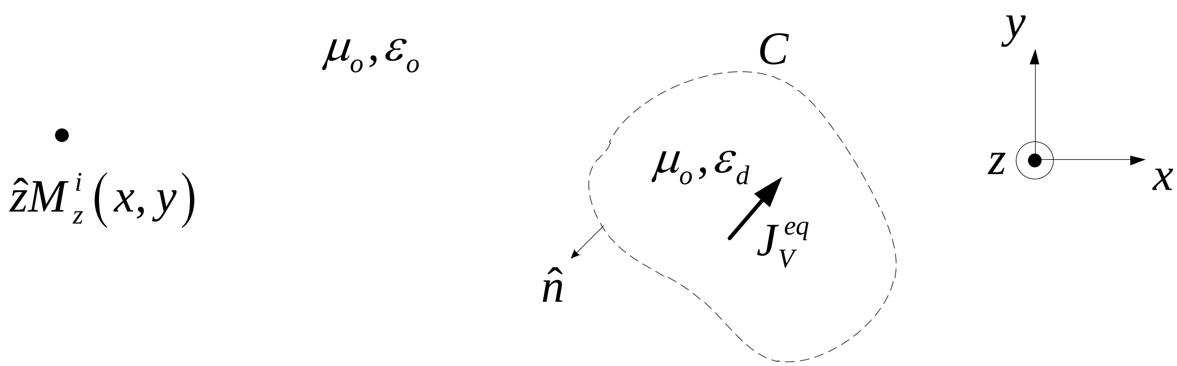
TE_z-Scattering by a Dielectric Cylinder – VEFIE Solution



- Consider a magnetic current density $\vec{M}^i(x,y) = \hat{z}M_z^i$ that is infinitely long and invariant w.r.t. the *z*-axis radiating in the presence of an infinitely long dielectric cylinder defined by a volume V_d that is situated in an infinite, homogeneous, unbounded media.
- Objective:
  - Compute the total fields inside and outside of the dielectric cylinder
- Solution
  - Pose the equivalent problem using volume equivalence
  - Derive the Volume Electric Field Integral Equation for TE-z polarization
  - Solve for the equivalent currents via the Method of Weighted Residuals
  - Compute the scattered fields from the equivalent currents

The Equivalent Problem

$$\left(\vec{E}^{inc}+\vec{E}^{scat},\vec{H}^{inc}+\vec{H}^{scat}\right)$$



- The dielectric is replaced with an electric volume equivalent current density that is radiating in a homogeneous free space:

$$\vec{J}_V^{eq}(\vec{r}) = j\omega\varepsilon_o(\varepsilon_r-1)\vec{E}^{tot}(\vec{r})$$

  where $\varepsilon_d=\varepsilon_r\varepsilon_o$
- Note that ε_r can be complex (lossy material), and can be inhomogeneous.
- We derived the VEFIE by enforcing $\vec{E}^{inc}(\vec{r})=\vec{E}^{tot}(\vec{r})-\vec{E}^{scat}(\vec{r}),\quad \vec{r}\in V_d$:
  - $$\vec{E}^{inc}(\vec{r})=\frac{\vec{J}_V^{eq}(\vec{r})}{j\omega\varepsilon_o(\varepsilon_r-1)}-\eta_o\frac{\nabla\nabla\cdot\vec{A}(\vec{r})+k_o^2\vec{A}(\vec{r})}{jk_o}$$
- where

$$\vec{A}(\vec{r})=\int_{S_d}\frac{1}{4j}H_0^{(2)}\left(k_o\left|\vec{r}-r'\right|\right)\vec{J}_V^{eq}(\vec{r}')ds'=\hat{x}A_x(\vec{r})+\hat{y}A_y(\vec{r})$$

- The $\nabla\nabla\cdot\vec{A}(\vec{r})$ can be expressed as:

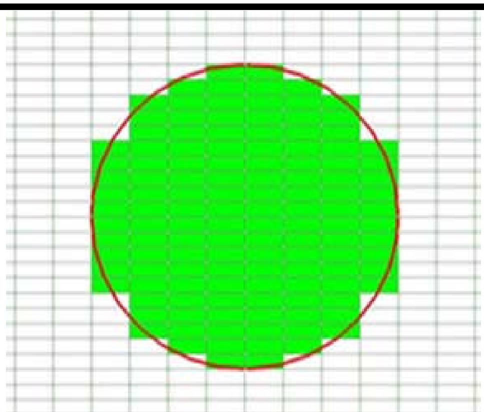
$$\nabla\nabla\cdot\vec{A}(\vec{r}) = \hat{x}\left(\frac{\partial^2 A_x(\vec{r})}{\partial x^2} + \frac{\partial^2 A_y(\vec{r})}{\partial x \partial y}\right) + \hat{y}\left(\frac{\partial^2 A_y(\vec{r})}{\partial y^2} + \frac{\partial^2 A_x(\vec{r})}{\partial y \partial x}\right)$$

- Then, breaking up the VEFIE into vector projections, it can be expressed as:

$$j\frac{k_o}{\eta_o}\begin{bmatrix} E_x^{inc}(\vec{r}) \\ E_y^{inc}(\vec{r}) \end{bmatrix} = \frac{1}{\varepsilon_r - 1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} J_{V_x}^{eq}(\vec{r}) \\ J_{V_y}^{eq}(\vec{r}) \end{bmatrix} - \begin{bmatrix} k_o^2 + \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & k_o^2 + \frac{\partial^2}{\partial y^2} \end{bmatrix}\begin{bmatrix} A_x(\vec{r}) \\ A_y(\vec{r}) \end{bmatrix}$$

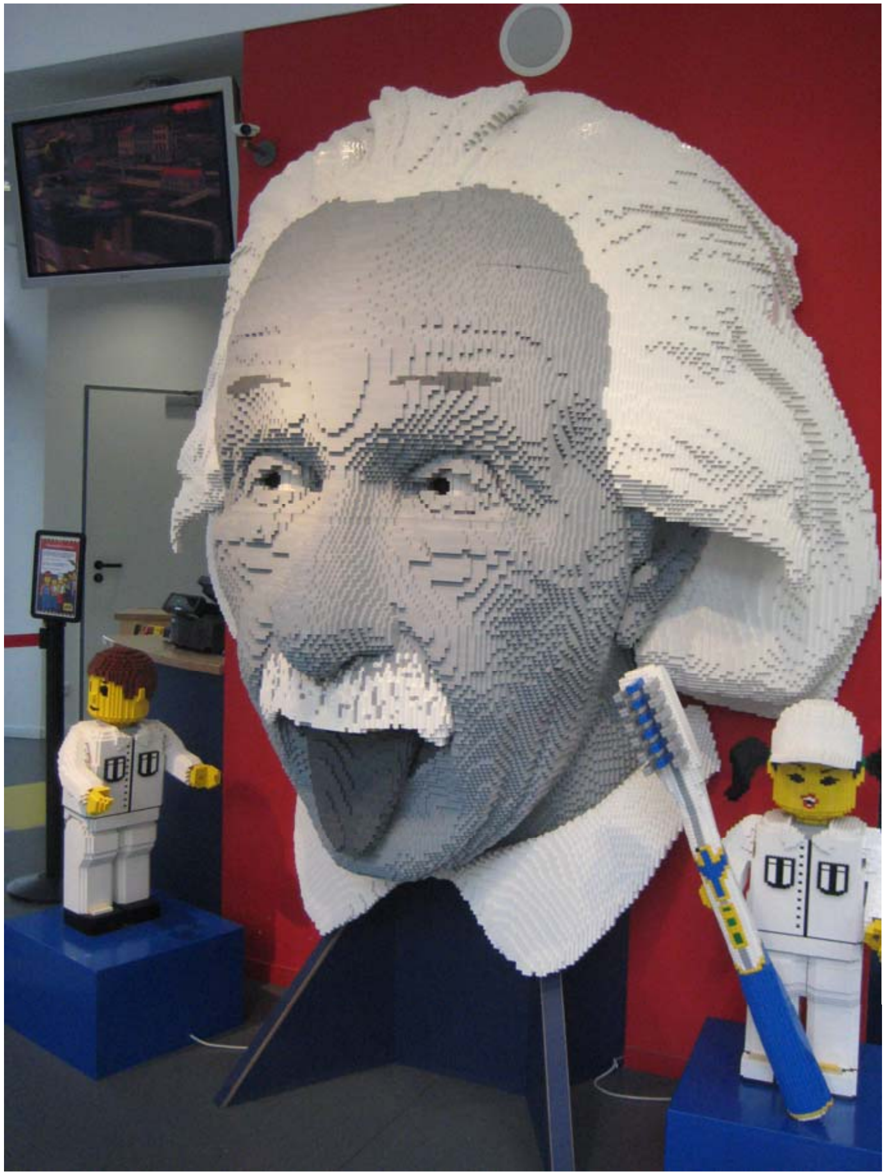
- Next, the MoM solution of the TEz-VEFIE will be pursued. To this end, $J_{V_x}^{eq}$ and $J_{V_y}^{eq}$ will be treated as unknowns.

The Method of Moment Solution



- Mesh
  - Rectangular cell approximation of the object
    - Note, this is a low-order approximation (simplest)
      - Triangles or arbitrary quads are more accurate
      - Curvilinear cells are most accurate
- Current basis function space:
  - Pulse approximation (piece-wise constant)
    - $J_{V_x}^{eq} \approx \sum_{n=1}^{N} \alpha_{x,n} P_n(\vec{r}), \quad J_{V_y}^{eq} \approx \sum_{n=1}^{N} \alpha_{y,n} P_n(\vec{r}), \qquad P_n(\vec{r}) = \begin{cases} 1, & \vec{r} \in S_n \\ 0, & \text{else} \end{cases}$
- Test function space:
  - Point-matching at the cell centers
    - $T_{x_m} = \delta\left(\left|\vec{r} - \vec{r}_m^{\,c}\right|\right), \quad T_{y_m} = \delta\left(\left|\vec{r} - \vec{r}_m^{\,c}\right|\right)$

Staircase-Approximation



Computing the Vector Potential

- Define
  - $A_{x,y}(\vec{r}) = \frac{1}{4j}\iint_S J^{eq}_{V_{x,y}}(\vec{r}) H_0^{(2)}(k_o R)ds' \approx \frac{1}{4j}\sum_{n=1}^{N}\alpha_{x,y,n}\iint_{S_n} H_0^{(2)}(k_o R)ds'$
- Again, use Richmond's approximation by approximating each rectangular cell as a circle with radius:
  - $a_n = \sqrt{\frac{\ell_n h_n}{\pi}}$
    - where ℓ_n and h_n are the dimensions of the cell
- The vector potential can be approximated as:
  - $A_{x,y}(\vec{r}) \approx \sum_{n=1}^{N}\alpha_{x,y,n}\frac{1}{4j}\int_0^{a_n}\int_0^{2\pi} H_0^{(2)}(k_o R)\rho' d\phi' d\rho'$
    - where $R = |\vec{r} - \vec{r}'|$

- To compute the integration, use the addition theorem:
  - $H_0^{(2)}\left(k_o\left|\vec{r}-\vec{r}'\right|\right)=\begin{cases}\sum\limits_{\nu=-\infty}^{\infty} J_\nu\left(k_o\rho'\right)H_\nu^{(2)}\left(k_o\rho\right)e^{j\nu(\phi-\phi')}, & \rho>\rho' \\ \sum\limits_{\nu=-\infty}^{\infty} J_\nu\left(k_o\rho\right)H_\nu^{(2)}\left(k_o\rho'\right)e^{j\nu(\phi-\phi')}, & \rho<\rho'\end{cases}$
    - where (ρ,ϕ) are the coordinates of $\vec{r}$, and (ρ',ϕ') of $\vec{r}'$
- For non-self-terms, the field point will be outside the source cell. Thus $\rho>\rho'$:
  - $\frac{1}{4j}\int\limits_0^{a_n}\int\limits_0^{2\pi} H_0^{(2)}\left(k_oR\right)\rho' d\phi' d\rho' = \frac{1}{4j}\int\limits_0^{a_n}\int\limits_0^{2\pi}\sum\limits_{\nu=-\infty}^{\infty} J_\nu\left(k_o\rho'\right)H_\nu^{(2)}\left(k_o\rho_n\right)e^{j\nu(\phi_m-\phi')}\rho' d\phi' d\rho'$
    - where ρ_n is the radial distance from $\vec{r}_n^{\,c}$ $\left(\rho_n=\sqrt{\left(x-x_n\right)^2+\left(y-y_n\right)^2}\right)$
- Note, that the ϕ-integration is only non-zero when $\nu=0$
  - $\frac{2\pi}{4j}H_0^{(2)}\left(k_o\rho_n\right)\int\limits_0^{a_n} J_0\left(k_o\rho'\right)\rho' d\phi' d\rho' = \frac{\pi}{2j}H_0^{(2)}\left(k_o\rho_n\right)\left(\frac{\rho'}{k_o}J_1\left(k_0\rho'\right)\Big|_0^{a_n}\right)$
  - $=\frac{\pi a_n}{2jk_o}H_0^{(2)}\left(k_o\rho_n\right)J_1\left(k_0a_n\right)$
- Therefore,
  - $A_{\substack{x,n\\y}}\left(\vec{r}\right)\approx\alpha_{\substack{x,n\\y}}\frac{\pi a_n}{2jk_o}J_1\left(k_0a_n\right)H_0^{(2)}\left(k_o\rho_n\right)$, for $\vec{r}\notin S_n$

- Recall, the VEFIE required derivatives of A in the observation coordinates
  - $\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right) A_{x,n}(x,y) \approx \alpha_{x,n} K_n \left(k_o^2 + \frac{\partial^2}{\partial x^2}\right) H_0^{(2)}(k_o \rho_n)$
    - where: $K_n = \frac{\pi a_n}{2jk_o} J_1(k_0 a_n)$
- Computing:
  - $\frac{\partial}{\partial x} H_0^{(2)}(k_o\rho_n) = k_o \frac{\partial}{\partial x}\rho_n H_0^{(2)\prime}(k_o\rho_n) = -k_o \frac{x-x_n}{\rho_n} H_1^{(2)}(k_o\rho_n)$
  - $$\frac{\partial^2}{\partial x^2} H_0^{(2)}(k_o\rho_n) = \frac{\partial}{\partial x}\left(-k_o \frac{x-x_n}{\rho_n} H_1^{(2)}(k_o\rho_n)\right)$$
    $$= -\frac{k_o}{\rho_n} H_1^{(2)}(k_o\rho_n) - k_o^2 \left(\frac{x-x_n}{\rho_n}\right)^2 H_0^{(2)}(k_o\rho_n) + 2k_o \frac{(x-x_n)^2}{\rho_n^3} H_1^{(2)}(k_o\rho_n)$$
  - Noting that:

$$-\frac{k_o}{\rho_n} H_1^{(2)}(k_o\rho_n) + 2k_o \frac{(x-x_n)^2}{\rho_n^3} H_1^{(2)}(k_o\rho_n) = -k_o \frac{\left(\rho_n^2 - 2(x-x_n)^2\right)}{\rho_n^3} H_1^{(2)}(k_o\rho_n)$$

$$= -k_o \frac{\left((y-y_n)^2 - (x-x_n)^2\right)}{\rho_n^3} H_1^{(2)}(k_o\rho_n)$$

- Also, noting that
  - $k_o^2 H_0^{(2)}(k_o\rho_n) - k_o^2 \frac{(x-x_n)^2}{\rho_n^2} H_0^{(2)}(k_o\rho_n) = k_o^2 \frac{(y-y_n)^2}{\rho_n^2} H_0^{(2)}(k_o\rho_n)$

- Combining these, finally leads to:

$$\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right) H_0^{(2)}(k_o\rho_n) = k_o^2 \frac{(y-y_n)^2}{\rho_n^2} H_0^{(2)}(k_o\rho_n) + k_o \frac{(x-x_n)^2 - (y-y_n)^2}{\rho_n^3} H_1^{(2)}(k_o\rho_n) = F_{x,x}(x,y)$$

- By duality:

$$\left(k_o^2 + \frac{\partial^2}{\partial y^2}\right) H_0^{(2)}(k_o\rho_n) = k_o^2 \frac{(x-x_n)^2}{\rho_n^2} H_0^{(2)}(k_o\rho_n) + k_o \frac{(y-y_n)^2 - (x-x_n)^2}{\rho_n^3} H_1^{(2)}(k_o\rho_n) = F_{y,y}(x,y)$$

- It can also be shown that:

$$\frac{\partial^2}{\partial x \partial y} H_0^{(2)}(k_o\rho_n) = k_o \frac{(y-y_n)(x-x_n)}{\rho_n^3} \left[2H_1^{(2)}(k_o\rho_n) - k_o\rho_n H_0^{(2)}(k_o\rho_n)\right] = F_{x,y}(x,y)$$

- where
  - $\rho_n = \sqrt{(x-x_n)^2 + (y-y_n)^2}$

The Impedance Matrix

- Applying point matching, the discrete VEFIE can be expressed as:

$$j\frac{k_o}{\eta_o}\begin{bmatrix} E_x^{inc}\left(\vec{r}_m\right) \\ E_y^{inc}\left(\vec{r}_m\right) \end{bmatrix} = \begin{bmatrix} \left[A_{m,n}\right] & \left[B_{m,n}\right] \\ \left[C_{m,n}\right] & \left[D_{m,n}\right] \end{bmatrix} \begin{bmatrix} \bar{\alpha}_x \\ \bar{\alpha}_y \end{bmatrix}$$

  - where, $\left[A_{m,n}\right]$, etc., are matrix blocks.

- For the non-self terms $(m \neq n)$,

$$A_{m,n} = -K_n F_{x,x}\left(x_m, y_m\right), \qquad B_{m,n} = -K_n F_{x,y}\left(x_m, y_m\right),$$

$$C_{m,n} = -K_n F_{x,y}\left(x_m, y_m\right), \qquad D_{m,n} = -K_n F_{y,y}\left(x_m, y_m\right).$$

- For example:
  - $A_{m,n} = -K_n\left[k_o^2\frac{\left(y_m - y_n\right)^2}{\rho_{m,n}^2}H_0^{(2)}\left(k_o\rho_{m,n}\right) + k_o\frac{\left(x_m - x_n\right)^2 - \left(y_m - y_n\right)^2}{\rho_{m,n}^3}H_1^{(2)}\left(k_o\rho_{m,n}\right)\right]$
  - where
    - $\rho_{m,n} = \sqrt{\left(x_m - x_n\right)^2 + \left(y_m - y_n\right)^2}$

Evaluating the Self Term

- Richmond's method is again used for the self-term:
  - $\frac{1}{4j}\iint_{S_n} H_0^{(2)}\left(k_o R_n\right) ds' \approx \frac{1}{4j}\int_0^{a_n}\int_0^{2\pi} H_0^{(2)}\left(k_o R_n\right)\rho' d\phi' d\rho'$
- Next, applying the addition theorem for $\rho < \rho'$
  - $\frac{1}{4j}\int_0^{a_n}\int_0^{2\pi} H_0^{(2)}\left(k_o R_n\right)\rho' d\phi' d\rho' = \frac{1}{4j}\left(\frac{2\pi a_n}{k_o} J_o\left(k_o\rho_n\right) H_1^{(2)}\left(k_o a_n\right) - j\frac{4}{k_o^2}\right)$
  - Note that ρ_n is the observation coordinate (distance from the center)
- Differentiating in the observation coordinates:
  - $\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right) J_0^{(2)}\left(k_o\rho_n\right) = k_o^2 \frac{\left(y-y_n\right)^2}{\rho_n^2} J_0\left(k_o\rho_n\right) + k_o \frac{\left(x-x_n\right)^2 - \left(y-y_n\right)^2}{\rho_n^3} J_1\left(k_o\rho_n\right)$
- For the self-term, this must be evaluated in the limit that $(x, y) \to (x_n, y_n)$
  - Note:
    - $\frac{\left(y-y_n\right)}{\rho_n} = \sin\phi_n,\ \frac{\left(x-x_n\right)}{\rho_n} = \cos\phi_n$
  - Therefore:
  - $\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right) J_0^{(2)}\left(k_o\rho_n\right) = k_o^2 \sin^2\phi_n J_0\left(k_o\rho_n\right) + k_o \frac{\left(\cos\phi_n\right)^2 - \left(\sin\phi_n\right)^2}{\rho_n} J_1\left(k_o\rho_n\right)$

- Now, take the limit as $\rho_n \to 0$

$$\lim_{\rho_n \to 0}\left[k_o^2 \sin^2\phi_n J_0\left(k_o\rho_n\right) + k_o \frac{\left(\cos\phi_n\right)^2 - \left(\sin\phi_n\right)^2}{\rho_n} J_1\left(k_o\rho_n\right)\right] =$$

$$= k_o^2 \sin^2\phi_n \cdot 1 + \left(\left(\cos\phi_n\right)^2 - \left(\sin\phi_n\right)^2\right)\frac{k_o}{\rho_n}\underbrace{\left(\frac{k_o\rho_n}{2}\right)}_{\substack{\text{small argument}\\ \text{approximation}}}$$

$$= \frac{k_o^2}{2}\left(\sin^2\phi_n + \cos^2\phi_n\right) = \frac{k_o^2}{2}$$

- For the constant term:
  - $\left(k_o^2 + \frac{\partial^2}{\partial x^2}\right)\frac{1}{4j}\left(-j\frac{4}{k_o^2}\right) = -1$
- Finally,
  - $A_{n,n} = \frac{1}{\varepsilon_{r_n} - 1} - \left(\frac{1}{4j}\frac{2\pi a_n}{k_o}\frac{k_o^2}{2}H_1^{(2)}\left(k_o a_n\right) - 1\right) = \frac{\varepsilon_{r_n}}{\varepsilon_{r_n} - 1} - \frac{\pi k_o a_n}{4j}H_1^{(2)}\left(k_o a_n\right)$
- By Duality,
  - $D_{n,n} = A_{n,n}$

- For the off-diagonal blocks,

$$\frac{\partial^2}{\partial x \partial_y} J_0^{(2)}\left(k_o\rho_n\right) = \frac{\left(x-x_n\right)\left(y-y_n\right)}{\rho_n^3}\left[2J_1\left(k_o\rho_n\right) - k_o\rho_n J_0\left(k_o\rho_n\right)\right]$$

$$= \frac{\cos\phi_n \sin\phi_n}{\rho_n}\left[2J_1\left(k_o\rho_n\right) - k_o\rho_n J_0\left(k_o\rho_n\right)\right]$$

  - $\lim_{\rho_n \to 0} \frac{\cos\phi_n \sin\phi_n}{\rho_n}\left[2J_1\left(k_o\rho_n\right) - k_o\rho_n J_0\left(k_o\rho_n\right)\right] = \frac{\cos\phi_n \sin\phi_n}{\rho_n}\left[2\frac{k_o\rho_n}{2} - k_o\rho_n \cdot 1\right] = 0$

- Therefore,
  - $B_{n,n} = C_{n,n} = 0$

The Impedance Matrix

- Summary:

$$j\frac{k_o}{\eta_o}\begin{bmatrix}\bar{e}_x\\ \bar{e}_y\end{bmatrix}=\begin{bmatrix}\left[A_{m,n}\right] & \left[B_{m,n}\right]\\ \left[C_{m,n}\right] & \left[D_{m,n}\right]\end{bmatrix}\begin{bmatrix}\bar{\alpha}_x\\ \bar{\alpha}_y\end{bmatrix}$$

$$A_{m,n}=-K_n\left[k_o^2\frac{\left(y_m-y_n\right)^2}{\rho_{m,n}^2}H_0^{(2)}\left(k_o\rho_{m,n}\right)+k_o\frac{\left(x_m-x_n\right)^2-\left(y_m-y_n\right)^2}{\rho_{m,n}^3}H_1^{(2)}\left(k_o\rho_{m,n}\right)\right],\ m\neq n$$

$$D_{m,n}=-K_n\left[k_o^2\frac{\left(x_m-x_n\right)^2}{\rho_{m,n}^2}H_0^{(2)}\left(k_o\rho_{m,n}\right)+k_o\frac{\left(y_m-y_n\right)^2-\left(x_m-x_n\right)^2}{\rho_{m,n}^3}H_1^{(2)}\left(k_o\rho_{m,n}\right)\right],\ m\neq n$$

$$C_{m,n}=B_{m,n}=-K_nk_o\frac{\left(y_m-y_n\right)\left(x_m-x_n\right)}{\rho_{m,n}^3}\left[2H_1^{(2)}\left(k_o\rho_{m,n}\right)-k_o\rho_nH_0^{(2)}\left(k_o\rho_{m,n}\right)\right],\ m\neq n$$

$$\text{where } K_n=\frac{\pi a_n}{2jk_o}J_1\left(k_0a_n\right)$$

$$A_{n,n}=D_{n,n}=\frac{\varepsilon_{r_n}}{\varepsilon_{r_n}-1}-\frac{\pi k_oa_n}{4j}H_1^{(2)}\left(k_oa_n\right),\quad C_{n,n}=B_{n,n}=0$$

$$e_{x_m}=jk_o\frac{E_o}{\eta_o}\sin\phi^{inc}e^{j\vec{k}^{inc}\cdot\vec{r}_m},\quad e_{y_m}=-jk_o\frac{E_o}{\eta_o}\cos\phi^{inc}e^{j\vec{k}^{inc}\cdot\vec{r}_m}$$

Far Field Approximation

- The scattered electric field is computed as:

$$E^{scat}(\vec{r}) = -jk_o\eta_o\vec{A}(\vec{r}) + \frac{\eta_o}{jk_o}\nabla\nabla\cdot\vec{A}(\vec{r})$$

- In the *far-field zone*

$$E^{scat}(\vec{r}) \approx -jk_o\eta_o\vec{A}(\vec{r})$$

  - For power propagation in the radiation direction, we are interested in E_ϕ^{scat}:
    - $E_\phi^{scat}(\vec{r}) \approx -jk_o\eta_o\left(-\sin\phi A_x(\vec{r}) + \cos\phi A_y(\vec{r})\right)$
    - where: $A_{\substack{x,n\\y}}(\vec{r}) \approx \sum_{n=1}^{N}\alpha_{\substack{x,n\\y}}\frac{\pi a_n}{2jk_o}J_1(k_1a_n)H_0^{(2)}(k_o\rho_n)$

- Far-field approximation:

$$\lim_{x\to\infty}H_0^{(2)}(k_o\rho_n) \approx \sqrt{\frac{2j}{\pi k_o\rho}}e^{-jk_o\rho}e^{jk_o\hat{\rho}\cdot\vec{r}_n}$$

  - where, $k_o\hat{\rho} = \hat{x}k_o\cos\phi + \hat{y}k_o\sin\phi = \vec{k}$

- Thus, in the far field:

$$E_\phi^{scat\,ff}(r,\phi) \approx \frac{\eta_o}{2}\sqrt{\frac{2j\pi}{k_o}}\frac{e^{-jk_o\rho}}{\sqrt{\rho}}\sum_{i=1}^{N}a_nJ_1(k_oa_n)\left[\alpha_{x_i}\sin\phi - \alpha_{y_i}\cos\phi\right]e^{j\vec{k}\cdot\vec{r}_n}$$

Echo Width

- Recall, the echo width of the target is:

$$\sigma_{2d}(\phi) = \lim_{k_o\rho\to\infty} 2\pi\rho \frac{P^s}{P^{inc}} = \lim_{k_o\rho\to\infty} 2\pi\rho \frac{\hat{r}\cdot\vec{E}^{scat}\times\vec{H}^{scat^*}}{\left|E^{inc}\right|^2/\eta_o} = \lim_{k_o\rho\to\infty} 2\pi\rho \frac{\left|E_\phi^{scat}\right|^2}{\left|E^{inc}\right|^2}$$

- This leads to:

$$\sigma_{2d}(\phi) = \frac{\eta_o^2\pi^2}{k_o E_o^2}\left|\sum_{i=1}^{N} a_n J_1(k_o a_n)\left[\alpha_{x_i}\sin\phi - \alpha_{y_i}\cos\phi\right]e^{j\vec{k}\cdot\vec{r}_n}\right|^2$$

  - where E_o is the amplitude of the incident electric field.
  - Or, since $H_o = E_o/\eta_o$

$$\sigma_{2d}(\phi) = \frac{\pi^2}{k_o H_o^2}\left|\sum_{i=1}^{N} a_n J_1(k_o a_n)\left[\alpha_{x_i}\sin\phi - \alpha_{y_i}\cos\phi\right]e^{j\vec{k}\cdot\vec{r}_n}\right|^2$$

MathCad Example

ka = 1 εr = 4 nx = 16 ny = 16 N = 861

Error = 0.035