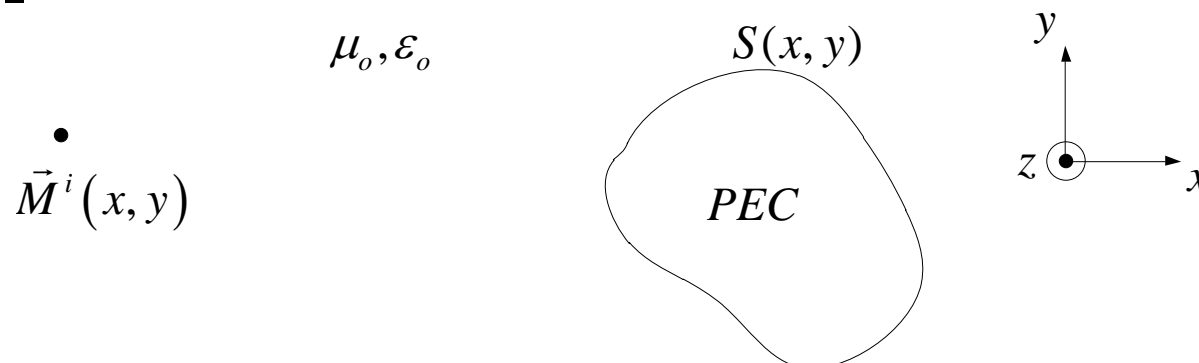
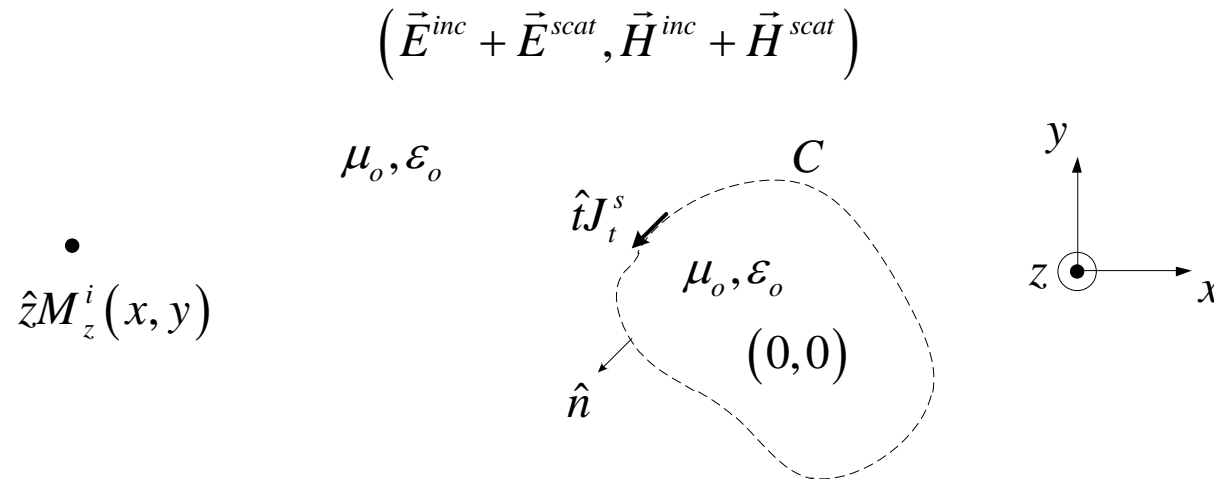


TE_z-Scattering by a PEC Cylinder – EFIE Solution



- Consider a magnetic current density $\vec{M}^i(x, y) = \hat{z}M_z^i$ that is infinitely long and invariant w.r.t. the z -axis radiating in the presence of an infinitely long PEC cylinder defined by the surface $S(x, y)$ which is situated in an infinite, homogeneous, unbounded media.
- Objective:
 - Compute the currents induced on the PEC cylinder
 - Compute the field scattered by the PEC Cylinder
- Solution
 - Pose the equivalent problem
 - Establish the Electric Field Integral Equation for this problem
 - Solve for the induced currents via the Method of Weighted Residuals
 - Compute the scattered fields from the induced currents

The Equivalent Problem



- Following Green's second identity, we can place an equivalent current density

$$\vec{J}_s = \hat{n} \times \vec{H}^{tot} \Big|_C = \hat{t} J_t^s$$

on S that is effectively radiating in a homogeneous free space.

- The equivalent current radiates the scattered field in the region exterior to C , and the negative of the incident field in the region interior to C .
- The electric field integral equation (EFIE) was derived by enforcing the physical boundary condition of the tangential electric field on the surface of the PEC:

$$\hat{n} \times \vec{E}^{tot} \Big|_C = 0,$$

or,

$$\hat{n} \times \vec{E}^{inc} \Big|_C = -\hat{n} \times \vec{E}^{scat} \Big|_C$$

- The scattered field is computed as:

$$\vec{E}^{scat}(\vec{r}) = -jk_o\eta_o\vec{A}(\vec{r}) + \frac{\eta_o}{jk_o}\nabla\nabla\cdot\vec{A}(\vec{r})$$

- where $k_o\eta_o = \omega\mu_o$, $\eta_o/k_o = 1/\omega\epsilon_o$, and

$$\vec{A}(\vec{r}) = \int_C \frac{1}{4j} H_0^{(2)}(k_o|\vec{r}-\vec{r}'|) \vec{J}_s(\vec{r}') d\ell'$$

- The boundary condition can be enforced by projecting the total field on the tangential vector directly. Thus, the EFIE can be written as:

$$\hat{t} \cdot \vec{E}^{inc}(\vec{r}) = jk_o\eta_o\hat{t} \cdot \vec{A}(\vec{r}) - \frac{\eta_o}{jk_o}\hat{t} \cdot \nabla\nabla\cdot\vec{A}(\vec{r}), \quad \vec{r} \in C$$

- Or, in terms of the scalar potential ($\Phi_e = -\frac{1}{j\omega\epsilon_o}\nabla\cdot\vec{A}$):

$$\hat{t} \cdot \vec{E}^{inc}(\vec{r}) = jk_o\eta_o\hat{t} \cdot \vec{A}(\vec{r}) + \hat{t} \cdot \nabla\Phi_e(\vec{r}), \quad \vec{r} \in C$$

The Method of Weighted Residuals Solution

$$\int_C \vec{T}(\vec{r}) \cdot \vec{E}^{inc}(\vec{r}) d\ell = \frac{k_o \eta_o}{4} \int_C \vec{T}(\vec{r}) \cdot \int_C \vec{J}_s(\vec{r}') H_0^{(2)}(k_o R) d\ell' + \int_C \vec{T}(\vec{r}) \cdot \nabla \Phi_e(\vec{r}) d\ell$$

- where $\Phi_e = -\frac{1}{j\omega\epsilon_o} \nabla \cdot \vec{A}$
- Basis functions interpolating the current must be:
 - tangential to C
 - sufficiently smooth
 - $\nabla \cdot \vec{A}$ implies $\nabla \cdot \vec{J}_s = -j\omega\rho$
 - $\nabla \cdot \vec{J}_s = \frac{\partial}{\partial \ell} J_t(\ell)$
 - If we use pulse basis functions for J , this would lead to singular charge
 - Option: choose smoother basis functions to represent J .
 - Triangular basis functions (linear)
- Test functions $\vec{T}(\vec{r})$ must be:
 - tangential to C
 - The $\vec{T}(\vec{r}) \cdot \nabla \Phi_e(\vec{r})$ implies additional smoothness needed on T
 - i.e., integration by parts leads to derivatives on T
 - Need a smoother representation of T
 - Pulse test functions.

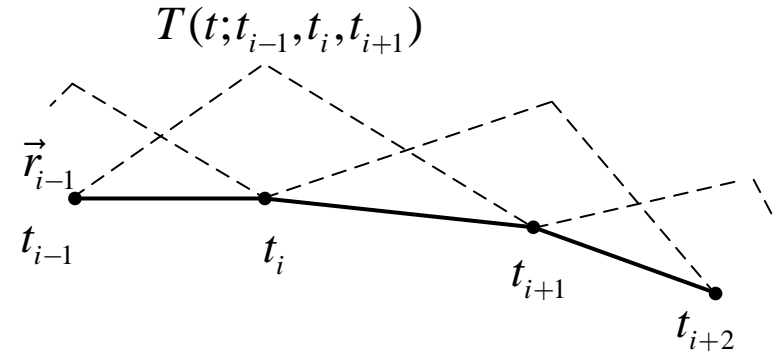
Discretization

- Triangular basis function expansion

- $\vec{J}_s \approx \sum_{i=1}^N \alpha_i T(t; t_{i-1}, t_i, t_{i+1}) \hat{t}$

- \hat{t} is the tangent to the contour

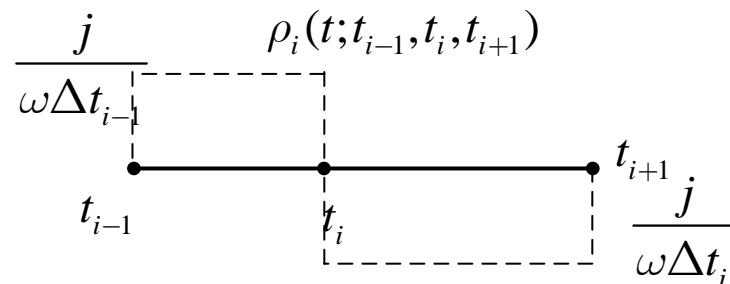
- $T(t; t_{i-1}, t_i, t_{i+1}) = \begin{cases} \frac{t - t_{i-1}}{\Delta t_{i-1}}, & t_{i-1} \leq t \leq t_i \\ \frac{t_{i+1} - t}{\Delta t_i}, & t_i \leq t \leq t_{i+1} \end{cases}$



- Charge density:

- $\rho = \frac{-1}{j\omega} \nabla \cdot \vec{J}_s \approx \frac{-1}{j\omega} \sum_{i=1}^N \alpha_i \frac{\partial}{\partial t} T(t; t_{i-1}, t_i, t_{i+1}) = \frac{-1}{j\omega} \begin{cases} \frac{1}{\Delta t_{i-1}}, & t_{i-1} \leq t \leq t_i \\ -\frac{1}{\Delta t_i}, & t_i \leq t \leq t_{i+1} \end{cases}$

- Charge doublet



- The charge density can also be written as a superposition of pulse functions:

$$\circ \rho \approx \frac{j}{\omega} \sum_{i=1}^N \alpha_i \left[\frac{1}{\Delta_{i-1}} P(t; t_{i-1}, t_i) - \frac{1}{\Delta_i} P(t; t_i, t_{i+1}) \right]$$

$$\blacksquare \text{ Pulse function: } P(t; t_i, t_{i+1}) = \begin{cases} 1, & t_i < t < t_{i+1} \\ 0, & \text{else} \end{cases}$$

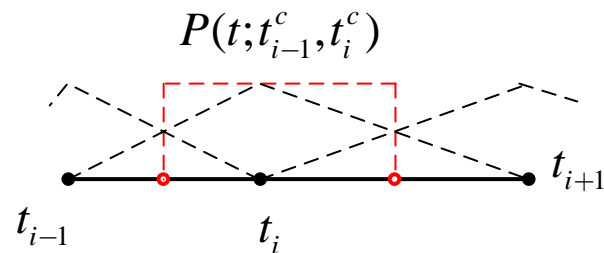
- Pulse test functions

- There is not sufficient smoothness to use delta-test functions

- Choose pulse functions

$$\blacksquare \vec{T}_i = \hat{t} P(t; t_{i-1}^c, t_i^c)$$

- Chosen to be centered about the current triangle



Evaluating the System Matrix – The Charge Term

- The electric scalar potential is expressed as:

$$\circ \Phi_e(\vec{r}) = \frac{1}{\epsilon_o} \int_C \rho(\vec{r}') \frac{1}{4j} H_0^{(2)}(k_o R) d\ell'$$

$$\blacksquare \text{ where } R = |\vec{r} - \vec{r}'|$$

- In the discrete space $\rho \approx \frac{j}{\omega} \sum_{i=1}^N \alpha_i \left[\frac{1}{\Delta_{i-1}} P(t; t_{i-1}, t_i) - \frac{1}{\Delta_i} P(t; t_i, t_{i+1}) \right]$. Therefore,

$$\circ \Phi_e(\vec{r}) = \frac{1}{4\omega\epsilon_o} \sum_{i=1}^N \alpha_i \left[\frac{1}{\Delta t_{i-1}} \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R) dt' - \frac{1}{\Delta t_i} \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R) dt' \right]$$

- Next, consider the outer integral due to the inner product:

$$\circ \int_C \vec{T}_j(\vec{r}) \cdot \nabla \Phi_e(\vec{r}) dt = \int_{t_{j-1}^c}^{t_j^c} \hat{t} \cdot \nabla \Phi_e(\vec{r}) dt = \int_{t_{j-1}^c}^{t_j^c} \frac{d}{dt} \Phi_e(\vec{r}) dt = \Phi_e(\vec{r}_j^c) - \Phi_e(\vec{r}_{j-1}^c)$$

- From above, this then becomes (where $R_j^c = |\vec{r}_j^c - \vec{r}'|$):

$$\circ \int_C \vec{T}_j(\vec{r}) \cdot \nabla \Phi_e(\vec{r}) dt = \frac{\eta_o}{4k_o} \sum_{i=1}^N \alpha_i \left[\frac{1}{\Delta t_{i-1}} \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_j^c) dt' - \frac{1}{\Delta t_i} \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_j^c) dt' - \frac{1}{\Delta t_{i-1}} \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_{j-1}^c) dt' + \frac{1}{\Delta t_i} \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \right]$$

Evaluating the System Matrix – The Vector Potential Term

- Next, consider the vector potential term:

$$\circ \int_C jk_o \eta_o \vec{T}(\vec{r}) \cdot \int_C \vec{J}_s(\vec{r}') \frac{1}{4j} H_0^{(2)}(k_o R) d\ell'$$

- In the discrete space, this becomes:

$$\circ \frac{k_o \eta_o}{4} \int_{t_{j-1}^c}^{t_j^c} \hat{t}(t) \cdot \sum_{i=1}^N \alpha_i \int_{t_{i-1}}^{t_{i+1}} \hat{t}(t') T(t'; t_{i-1}, t_i, t_{i+1}) H_0^{(2)}(k_o R) dt'$$

- To simplify this, we can approximation the integral over the triangular pulse function with the equivalent moment of a pulse function:

$$\circ \int_{t_{i-1}}^{t_{i+1}} T(t'; t_{i-1}, t_i, t_{i+1}) H_0^{(2)}(k_o R) dt' \approx \int_{t_{i-1}^c}^{t_i^c} H_0^{(2)}(k_o R) dt'$$

- [see A.W. Glisson & D.R. Wilton, IEEE T. Ant. Prop., Sept. 1980]

- Next, we can approximate the exterior integral with a mid-point rule:

$$\circ \frac{k_o \eta_o}{4} \int_{t_{j-1}^c}^{t_j^c} \hat{t}(t) \cdot \sum_{i=1}^N \alpha_i \int_{t_{i-1}}^{t_{i+1}} \hat{t}(t') T(t'; t_{i-1}, t_i, t_{i+1}) H_0^{(2)}(k_o R) dt' \approx$$

$$\frac{k_o \eta_o}{4} (\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c)) \cdot \sum_{i=1}^N \alpha_i \int_{t_{i-1}^c}^{t_i^c} \hat{t} H_0^{(2)}(k_o R_j) dt'$$

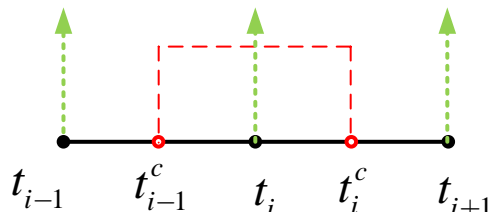
- where $\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c)$ approximates $(\hat{t}_j \Delta t_j + \hat{t}_j \Delta t_{j-1}) / 2$

Evaluating the System Matrix

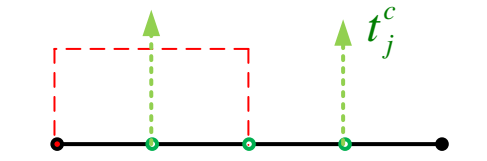
- In summary, each element of the system matrix is approximated as:

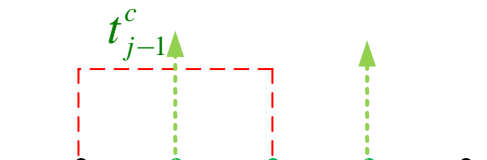
$$Z_{j,i} = \frac{k_o \eta_o}{4} (\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c)) \cdot \int_{t_{i-1}^c}^{t_i^c} \hat{t} H_0^{(2)}(k_o R_j) dt' + \frac{\eta_o}{4k_o} \left[\frac{1}{\Delta t_{i-1}} \left(\int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_j^c) dt' - \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) - \frac{1}{\Delta t_i} \left(\int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_j^c) dt' - \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) \right]$$


$Z_{j,i}$ consists of 5 integrals:

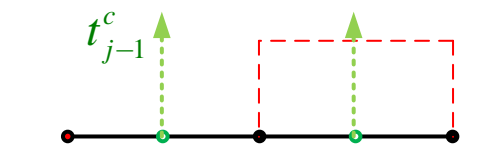
$$1. \int_{t_{i-1}^c}^{t_i^c} H_0^{(2)}(k_o R_j) dt'$$


Singular when $j = i$

2. $\int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_j^c) dt'$  Singular when $j = i-1$

3. $\int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_{j-1}^c) dt'$  Singular when $j = i$

4. $\int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_j^c) dt'$  Singular when $j = i$

5. $\int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt'$  Singular when $j = i+1$

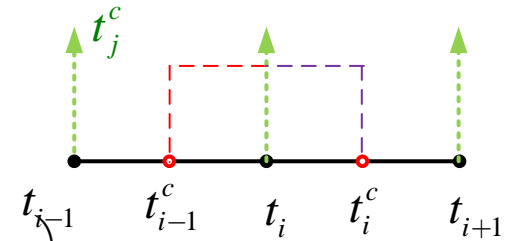
Evaluating the Non-Singular Terms

A.
$$\frac{k_o \eta_o}{4} \left(\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c) \right) \cdot \int_{t_{i-1}^c}^{t_i^c} \hat{t} H_0^{(2)}(k_o R_j) dt' \quad , (j \neq i)$$

- Break up into two pieces:

- $$= \frac{k_o \eta_o}{4} \left(\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c) \right) \cdot \left(\int_{t_{i-1}^c}^{t_i} \hat{t}_{i-1} H_0^{(2)}(k_o R_j) dt' + \int_{t_i}^{t_i^c} \hat{t}_i H_0^{(2)}(k_o R_j) dt' \right)$$

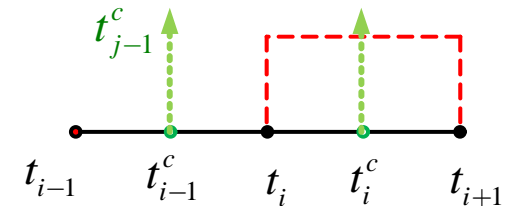
- Use Gauss-Legendre (3 point rule is probably sufficient) over each 1/2 segment.



B.
$$\frac{\eta_o}{4k_o} \frac{1}{\Delta t_{i-1}} \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \quad , (j \neq i+1)$$

- Can integrate over the entire interval:

- Use Gauss-Legendre integration rule (3 point rule is probably sufficient)



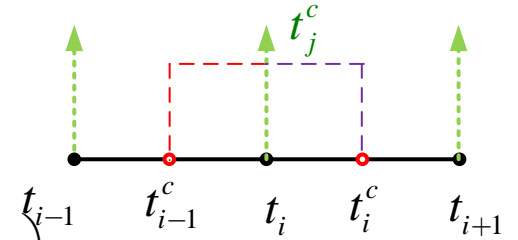
Computing the Self Term

A.
$$\frac{k_o \eta_o}{4} (\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c)) \cdot \int_{t_{i-1}^c}^{t_i^c} \hat{t} H_0^{(2)}(k_o R_j) dt' \quad (j = i)$$

- Again, break up into two pieces:

○
$$= \frac{k_o \eta_o}{4} (\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c)) \cdot \left(\int_{t_{i-1}^c}^{t_i} \hat{t}_{i-1} H_0^{(2)}(k_o R_i) dt' + \int_{t_i}^{t_i^c} \hat{t}_i H_0^{(2)}(k_o R_i) dt' \right)$$

- Use Gauss-lin-log rule over each 1/2 segment.



B.
$$\frac{\eta_o}{4k_o \Delta t_{i-1}} \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \quad , \quad (j = i + 1)$$

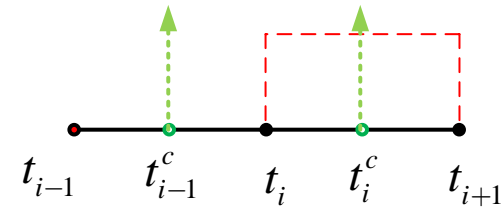
- Break into two parts:

○
$$\frac{\eta_o}{4k_o \Delta t_{i-1}} \left(\int_{t_i}^{t_i^c} H_0^{(2)}(k_o R_{j-1}^c) dt' + \int_{t_i^c}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) = \frac{\eta_o}{2k_o \Delta t_{i-1}} \int_{t_i^c}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt'$$

- Use Gauss-lin-log integration rule over the 1/2 segment

- Finally, note that:

○
$$\int_{t_i}^{t_i^c} H_0^{(2)}(k_o R_i^c) dt' = \int_{t_i^c}^{t_{i+1}} H_0^{(2)}(k_o R_i^c) dt' = \int_{t_i}^{t_i^c} H_0^{(2)}(k_o R_i) dt'$$



Logistics

$$Z_{j,i} = \frac{k_o \eta_o}{4} \left(\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c) \right) \cdot \int_{t_{i-1}^c}^{t_i^c} \hat{t} H_0^{(2)}(k_o R_j) dt' + \frac{\eta_o}{4k_o} \left[\begin{array}{l} \frac{1}{\Delta t_{i-1}} \left(\int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_j^c) dt' - \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) \\ - \frac{1}{\Delta t_i} \left(\int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_j^c) dt' - \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) \end{array} \right]$$

- Let:

- $I_{j,i}^{\Phi 1} = \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_j^c) dt'$, $I_{j,i}^{\Phi 2} = \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_{j-1}^c) dt'$, $I_{j,i}^{\Phi 3} = \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_j^c) dt'$, $I_{j,i}^{\Phi 4} = \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt'$

- It is observed that:

- $I_{j+1,i}^{\Phi 2} = I_{j,i}^{\Phi 1}$, $I_{j,i-1}^{\Phi 3} = I_{j,i}^{\Phi 1}$, $I_{j+1,i-1}^{\Phi 4} = I_{j,i}^{\Phi 1}$

- Therefore, you can compute one integral ($I_{j,i}^{\Phi 1}$), and have it contribute to four different impedance elements ($Z_{j,i}, Z_{j+1,i}, Z_{j,i-1}, Z_{j+1,i-1}$)

- Care needs to be taken

- Closed objects, used modulo arithmetic
 - Assumes continuous ordering of the linear segments
- Open objects
 - Terms that go beyond edges are not needed

Method of Moment Linear System

- The forcing vector is finally computed using a mid-point rule

$$\circ \int_C \vec{T}_j(\vec{r}) \cdot \vec{E}^{inc}(\vec{r}) d\ell \approx \left(\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c) \right) \cdot \vec{E}^{inc}(\vec{r}_j)$$

- Summary:

$$Z_{j,i} = \frac{k_o \eta_o}{4} \left(\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c) \right) \cdot \int_{t_{i-1}^c}^{t_i^c} \hat{t} H_0^{(2)}(k_o R_j) dt' + \frac{\eta_o}{4k_o} \left[\begin{array}{l} \frac{1}{\Delta t_{i-1}} \left(\int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_j^c) dt' - \int_{t_{i-1}}^{t_i} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) \\ - \frac{1}{\Delta t_i} \left(\int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_j^c) dt' - \int_{t_i}^{t_{i+1}} H_0^{(2)}(k_o R_{j-1}^c) dt' \right) \end{array} \right]$$

$$f_j = \left(\vec{r}(t_j^c) - \vec{r}(t_{j-1}^c) \right) \cdot \vec{E}^{inc}(\vec{r}_j)$$

- Solve:

$$\bar{\bar{Z}} \bar{\alpha} = \bar{f}$$

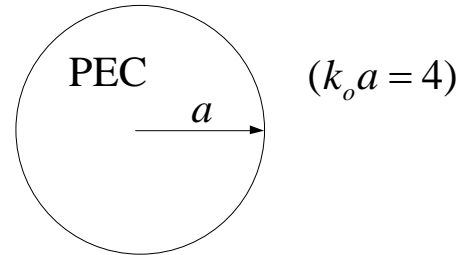
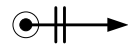
- Once the solution vector is found:

$$\vec{J}_s \approx \sum_{i=1}^N \alpha_i T(t; t_{i-1}, t_i, t_{i+1}) \hat{t}$$

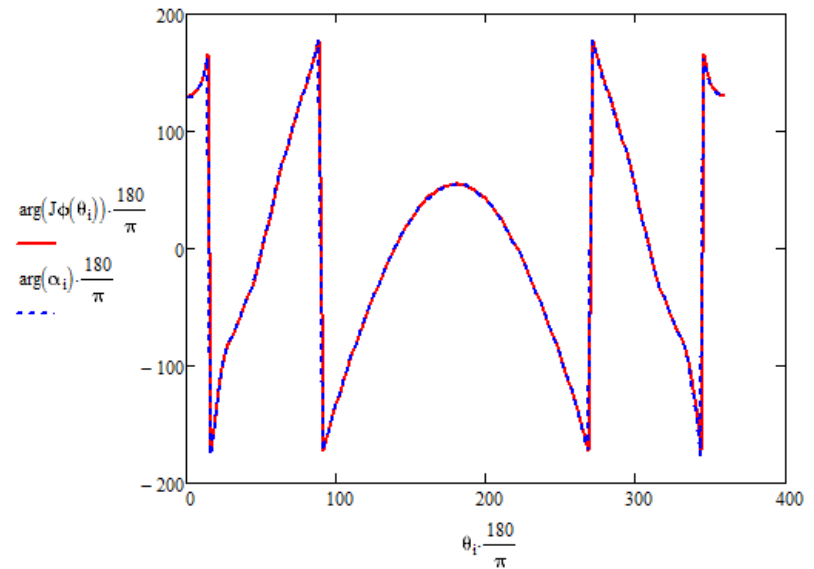
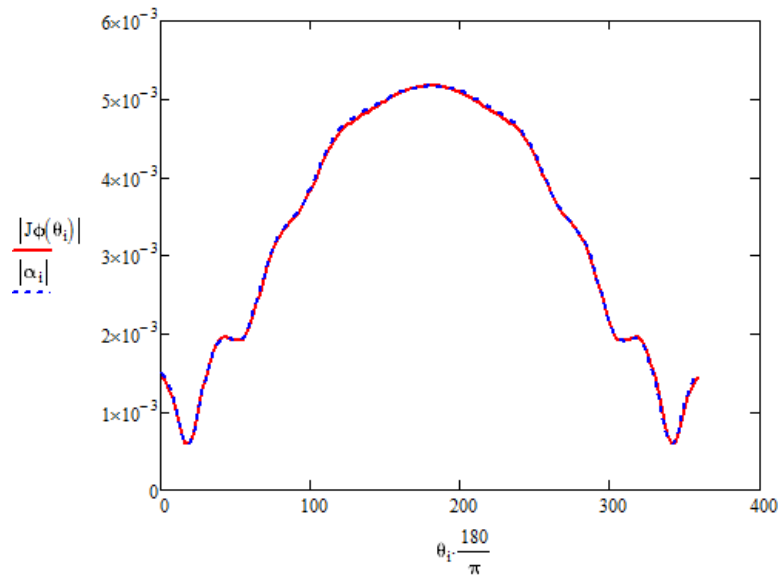
• **Example: Scattering of a TE_z Plane Wave by a Circular Cylinder**

μ_o, ϵ_o

$\vec{E}^{inc} = \hat{y}e^{-jk_o x}$ V/m



- Piece-wise linear approximation of the surface
 - EFIE formulation
 - MoM solution with Pulse basis functions and Point-Matching
 - Induced current density (Exact (J_ϕ), vs. MoM (α))
 - 160 uniformly spaced segments



TEz Scattering by a Circular Cylinder - MoM-EFIE Solution

$$ka := 4 \quad a := \frac{ka}{2\pi} \quad ko := 2\pi$$

$$N := 160$$

$$dp := \frac{2\pi}{N} \quad p(i) := i \cdot dp \quad pc(i) := p(i) + \frac{dp}{2} \quad \eta := 376.7303134617706554679$$

$$nc(i) := \begin{pmatrix} \cos(pc(i)) \\ \sin(pc(i)) \\ 0 \end{pmatrix} \quad tc(i) := \begin{pmatrix} -\sin(pc(i)) \\ \cos(pc(i)) \\ 0 \end{pmatrix} \quad rc(i) := nc(i) \cdot a$$

$$g := 0.57721566490153286060651209008240243104215933593992$$

$$dl := a \cdot dp \quad H02(x) := J0(x) - j \cdot Y0(x) \quad \gamma := e^g = 1.781072417990198$$

$$rp(i, x) := rc(i) + tc(i) \cdot (x - 0.5) \cdot dl$$

$$rphp(i, x) := rc(i) + tc(i) \cdot x \cdot \frac{dl}{2} \quad rphm(i, x) := rc(i) - tc(i) \cdot (1 - x) \cdot \frac{dl}{2}$$

$$K1 := \frac{ko \cdot \eta \cdot dl}{4} \quad kv := \begin{pmatrix} ko \\ 0 \\ 0 \end{pmatrix} \quad Einc(i) := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot e^{-j \cdot kv \cdot rp(i, 0)}$$

$$K2 := \frac{\eta}{4 \cdot ko}$$

$$NqLLm1 := 2$$

$$xll_0 := 0.02881166253095182700 \quad wll_0 := 0.1033307079649286500$$

$$xll_1 := 0.3040637296121376200 \quad wll_1 := 0.4546365259700986200$$

$$xll_2 := 0.8116692253440781200 \quad wll_2 := 0.4420327660649726600$$

$$NqGLm1 := 2$$

$$xgl_0 := 0.1127016653792583114820 \quad wgl_0 := 0.27777777777777777777777777777780$$

$$xgl_1 := 0.5 \quad wgl_1 := 0.44444444444444444444444444444440$$

$$xgl_2 := 1 - xgl_0 \quad wgl_2 := wgl_0$$

$$cycle(i) := \begin{cases} (N + i) & \text{if } i < 0 \\ (i - N) & \text{if } i \geq N \\ i & \text{otherwise} \end{cases}$$

$$Zself := \sum_{k=0}^{NqLLm1} \left(\frac{wll_k}{2} \cdot H02\left(ko \cdot xll_k \cdot \frac{dl}{2}\right) \right) \quad \% \text{ Compute 1X this integral due to uniformity of mesh}$$

$$Zself = 0.5 + 1.164i$$

$$intAm(j, ii) := \begin{cases} j \leftarrow cycle(jj) \\ i \leftarrow cycle(ii) \\ im1 \leftarrow cycle(ii - 1) \\ jm1 \leftarrow cycle(jj - 1) \\ \left[\left[(rc(j) - rc(jm1)) \cdot tc(im1) \cdot \sum_{k=0}^{NqGLm1} \left(\frac{wgl_k}{2} \cdot H02(ko \cdot |rp(j, 0) - rphp(im1, xgl_k)|) \right) \right] \right] & \text{if } i \neq j \\ \left[\left[(rc(j) - rc(jm1)) \cdot tc(im1) \right] \cdot Zself \right] & \text{otherwise} \end{cases}$$

$$intAp(j, ii) := \begin{cases} j \leftarrow cycle(jj) \\ i \leftarrow cycle(ii) \\ jm1 \leftarrow cycle(jj - 1) \\ \left[\left[(rc(j) - rc(jm1)) \cdot tc(i) \cdot \sum_{k=0}^{NqGLm1} \left(\frac{wgl_k}{2} \cdot H02(ko \cdot |rp(j, 0) - rphm(i, xgl_k)|) \right) \right] \right] & \text{if } i \neq j \\ \left[\left[(rc(j) - rc(jm1)) \cdot tc(i) \right] \cdot Zself \right] & \text{otherwise} \end{cases}$$

$$intP(j, ii) := \begin{cases} j \leftarrow cycle(jj) \\ i \leftarrow cycle(ii) \\ NqGLm1 \\ \sum_{k=0}^{NqGLm1} \left(wgl_k \cdot H02(ko \cdot |rc(j) - rp(i, xgl_k)|) \right) & \text{if } i \neq j \\ 2 \cdot Zself & \text{otherwise} \end{cases}$$

$$Z := \begin{cases} \text{for } j \in 0..N - 1 \\ \quad \text{for } i \in 0..N - 1 \\ \quad \quad Av \leftarrow K1 \cdot (intAp(j, i) + intAm(j, i)) \\ \quad \quad Phie \leftarrow K2 \cdot (intP(j, i - 1) - intP(j - 1, i - 1) - intP(j, i) + intP(j - 1, i)) \\ \quad \quad tmp_{j,i} \leftarrow Av + Phie \\ \quad tmp \end{cases}$$

$$fe := \begin{cases} \text{for } j \in 0..N - 1 \\ \quad jm1 \leftarrow cycle(j - 1) \\ \quad tmp_j \leftarrow (rc(j) - rc(jm1)) \cdot \overline{Einc(j)} \\ \quad tmp \end{cases}$$

$$Z = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.185-30.915i & 0.184+9.062i & 0.182+3.222i \\ 1 & 0.184+9.062i & 0.185-30.915i & 0.184+9.062i \\ 2 & 0.182+3.222i & 0.184+9.062i & \dots \end{array}$$

$$\alpha := Z^{-1} \cdot fe$$

$$\theta := \begin{cases} \text{for } j \in 0..N-1 \\ \text{tmp}_j \leftarrow pc(j) \\ \text{tmp} \end{cases}$$

$$Jnp(n,x) := Jn(n-1,x) - \frac{n}{x} \cdot Jn(n,x)$$

$$Hn2p(n,x) := Jnp(n,x) - j \cdot Ynp(n,x)$$

$$Hn2(n,x) := Jn(n,x) - j \cdot Yn(n,x)$$

$$Nmax := \begin{cases} \text{ceil}(6 \cdot ka) & \text{if } \text{ceil}(6 \cdot ka) > 10 \\ 10 & \text{otherwise} \end{cases}$$

$$J\phi(\phi) := \frac{2j}{\pi \cdot ka \cdot \eta} \sum_{n=0}^{Nmax} \left[(j)^{-n} \cdot \kappa(n) \cdot \frac{\cos(n \cdot \phi)}{Hn2p(n,ka)} \right]$$

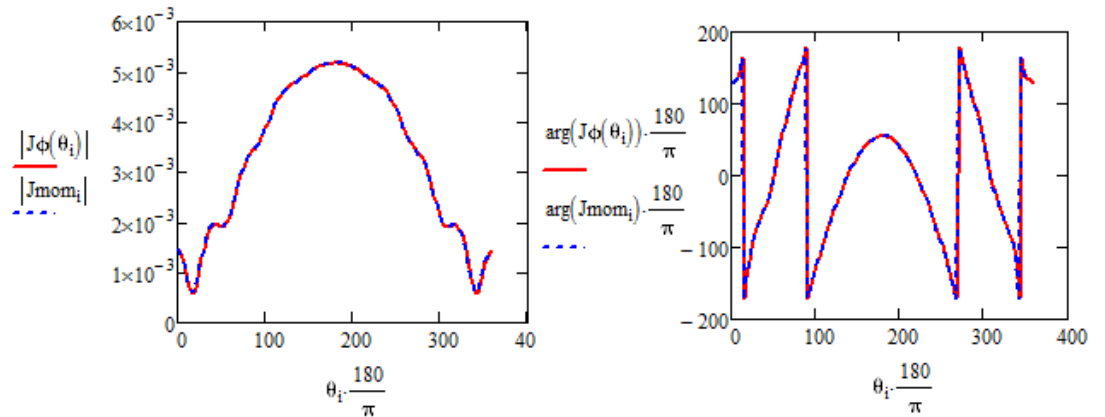
$$\kappa(n) := \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{otherwise} \end{cases}$$

$$fe = \begin{array}{c|c} & 0 \\ \hline 0 & -0.016+0.019i \\ 1 & -0.016+0.019i \\ 2 & -0.017+0.019i \\ 3 & -0.017+0.018i \\ 4 & \dots \end{array}$$

$$Ynp(n,x) := Yn(n-1,x) - \frac{n}{x} \cdot Yn(n,x)$$

$$i := 0..N-1$$

$$Jmom_i := \begin{cases} ip1 \leftarrow \text{cycle}(i+1) \\ \frac{1}{2} \cdot (\alpha_i + \alpha_{ip1}) \end{cases}$$



$$\text{Error} := \frac{1}{N} \sum_{k=0}^{N-1} \frac{(|J\phi(\theta_k) - Jmom_k|)}{|J\phi(\theta_k)|}$$

Error = 6.065 × 10⁻³

Far Field Approximation

- The scattered electric field is computed as:

$$E^{scat}(\vec{r}) = -jk_o\eta_o\vec{A}(\vec{r}) + \frac{\eta_o}{jk_o}\nabla\nabla\cdot\vec{A}(\vec{r})$$

- In the *far-field zone*

$$E^{scat}(\vec{r}) \approx -jk_o\eta_o\vec{A}(\vec{r})$$

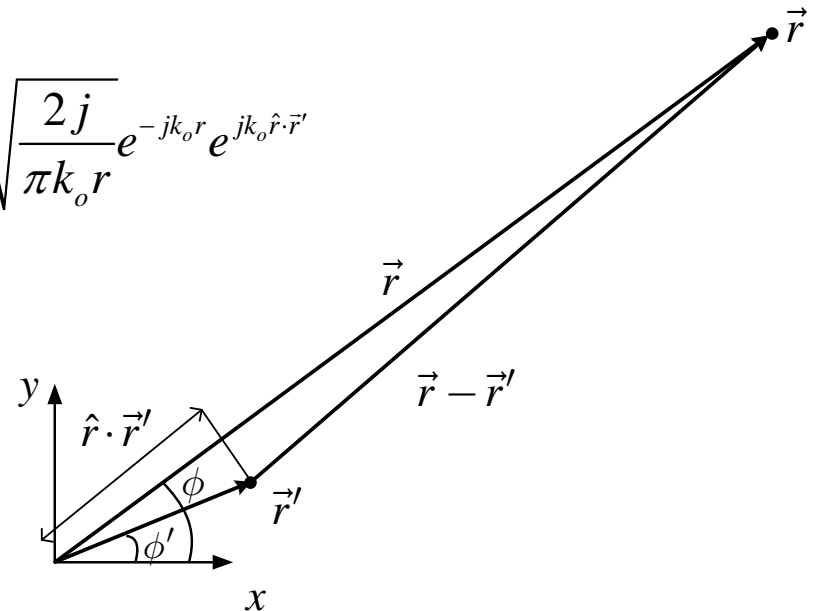
- The gradient divergence term will not radiate into the far field, b/c of $1/\rho$ decay.
- For power propagation in the radiation direction, we are interested in E_ϕ^{scat} :
 - $E^{scat}(\vec{r}) \approx -jk_o\eta_o(-\sin\phi A_x(\vec{r}) + \cos\phi A_y(\vec{r}))$

- Large argument & far-field approximations:

$$\lim_{x \rightarrow \infty} H_0^{(2)}(k_o R) \approx \sqrt{\frac{2j}{\pi k_o R}} e^{-jk_o R} \approx \sqrt{\frac{2j}{\pi k_o r}} e^{-jk_o r} e^{jk_o \hat{r} \cdot \vec{r}'}$$

- where,

- $\hat{r} = \hat{x} \cos\phi + \hat{y} \sin\phi$



- Thus, in the far field:

$$E_{\phi}^{scat^{ff}}(r, \phi) \approx -\eta \sqrt{\frac{jk_o}{8\pi}} \frac{e^{-jk_o r}}{\sqrt{r}} \sum_{i=1}^N \alpha_i \int_{t_{i-1}}^{t_{i+1}} (-t_x(t') \sin \phi + t_y(t') \cos \phi) T(t', t_{i-1}, t_i, t_{i+1}) e^{jk_o(x(t') \cos \phi + y(t') \sin \phi)} dt'$$

- Approximate using a mid-point rule:

$$E_{\phi}^{scat^{ff}}(r, \phi) \approx -\eta \sqrt{\frac{jk_o}{8\pi}} \frac{e^{-jk_o r}}{\sqrt{r}} \sum_{i=1}^N \frac{\alpha_i + \alpha_{i+1 \% N}}{2} (-t_{x_i} \sin \phi + t_{y_i} \cos \phi) e^{jk_o(x_i^c \cos \phi + y_i^c \sin \phi)}$$

Echo Width

- Recall, the echo width of the target is:

$$\sigma_{2d}(\phi) = \lim_{k_o r \rightarrow \infty} 2\pi r \frac{P^s}{P^{inc}} = \lim_{k_o r \rightarrow \infty} 2\pi r \frac{\hat{r} \cdot \vec{E}^{scat} \times \vec{H}^{scat*}}{|E^{inc}|^2 / \eta_o} = \lim_{k_o r \rightarrow \infty} 2\pi r \frac{|E_{\phi}^{scat}|^2}{|E^{inc}|^2}$$

- Given

$$E_{\phi}^{scat,ff}(r, \phi) \approx -\eta_o \sqrt{\frac{jk_o}{8\pi}} \frac{e^{-jk_o r}}{\sqrt{r}} \sum_{i=1}^N \alpha_i \left[\int_{t_{i-1}}^{t_i} (-t_{x_{i-1}} \sin \phi + t_{y_{i-1}} \cos \phi) \frac{t' - t_{i-1}}{\Delta t_{i-1}} e^{jk_o(x(t') \cos \phi + y(t') \sin \phi)} dt' \right. \\ \left. + \int_{t_i}^{t_{i+1}} (-t_{x_i} \sin \phi + t_{y_i} \cos \phi) \frac{t_{i+1} - t'}{\Delta t_i} e^{jk_o(x(t') \cos \phi + y(t') \sin \phi)} dt' \right]$$

- This leads to:

$$\sigma_{2d}(\phi) = \frac{k_o \eta_o^2}{4 E_o^2} \left| \sum_{i=1}^N \frac{\alpha_i + \alpha_{i+1 \% N}}{2} (-t_{x_i} \sin \phi + t_{y_i} \cos \phi) e^{jk_o(x_i^c \cos \phi + y_i^c \sin \phi)} \right|^2$$

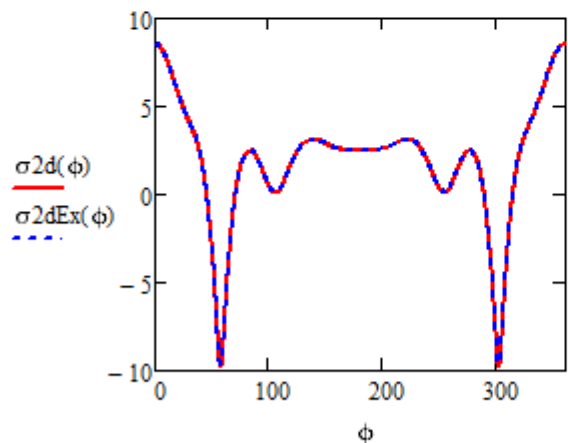
- o where E_o is the amplitude of the incident electric field.

$ka = 4, N = 160$:

$$\mathbf{fhat}(\phi) := \begin{pmatrix} -\sin\left(\phi \cdot \frac{\pi}{180}\right) \\ \cos\left(\phi \cdot \frac{\pi}{180}\right) \\ 0 \end{pmatrix} \quad \mathbf{rhat}(\phi) := \begin{pmatrix} \cos\left(\phi \cdot \frac{\pi}{180}\right) \\ \sin\left(\phi \cdot \frac{\pi}{180}\right) \\ 0 \end{pmatrix}$$

$$\sigma_{2d}(\phi) := 10 \log \left[\frac{ko \cdot \eta^2 \cdot dl^2}{4} \cdot \left| \sum_{k=0}^{N-1} \left[(\mathbf{fhat}(\phi) \cdot \mathbf{tc}(k)) \cdot \mathbf{Jmom}_k \cdot e^{j \cdot ko \cdot \mathbf{rc}(k) \cdot \mathbf{rhat}(\phi)} \right] \right|^2 \right]$$

$$\sigma_{2dEx}(\phi) := 10 \log \left[\frac{4}{ko} \cdot \left| \sum_{n=0}^{Nmax} \left(\kappa(n) \cdot \frac{\mathbf{Jnp}(n, ka) \cdot \cos\left(n \cdot \phi \cdot \frac{\pi}{180}\right)}{\mathbf{Hn2p}(n, ka)} \right) \right|^2 \right]$$



$$\text{Error}\sigma := \frac{1}{N} \cdot \sum_{k=0}^{N-1} \frac{(|\sigma_{2d}(\theta_k) - \sigma_{2dEx}(\theta_k)|)}{|\sigma_{2d}(\theta_k)|}$$

$$\text{Error}\sigma = 2.168 \times 10^{-3}$$