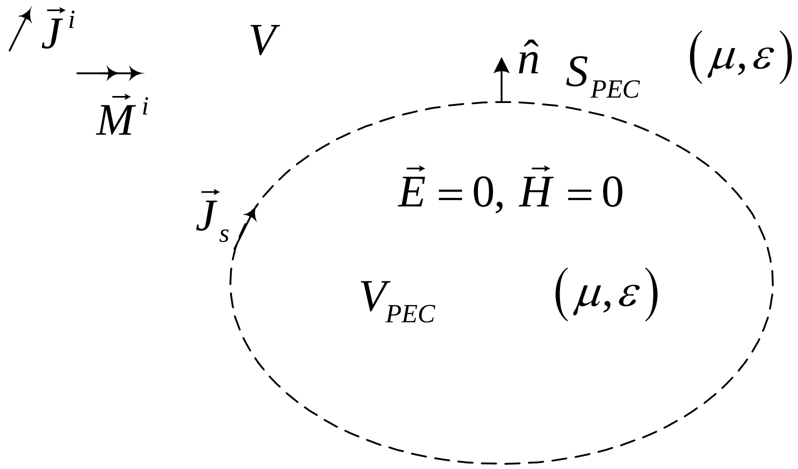
Solving for the Surface Current Density



- The EFIE, MFIE, or CFIE can be posed to allow for the solution of the surface current density.
- For example, the EFIE is:
  - $-\hat{n}\times\vec{E}^{i}(\mathbf{r})=-\hat{n}\times j\omega\mu\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'+\hat{n}\times\frac{1}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$
    - where, $\mathbf{r}\in S_{PEC}$
- The next objective is to explore *how* one can go about solving for the unknown current density.
- This will be done via the *Method of Weighted Residuals*, better known in the CEM community as the *Method of Moments (MoM)*.

Representing the EFIE as a Linear Operator

- The integral equations can be written in the form as: $f = \mathfrak{L}x$
  - o f = the forcing function (known excitation)
  - o $\mathfrak{L}$ = the integral operator (Assumed to be a linear operator)
  - o x = the unknown function (current density) distributed over S

- For example, consider the EFIE:
  - o $\vec{E}^i_{\tan}(\mathbf{r}) = \vec{E}^s_{\tan}(\mathbf{r}), \quad \mathbf{r} \in S_{PEC}$
    - ▪ More explicitly:
  - o $\vec{E}^i_{\tan}(\mathbf{r}) = j\omega\mu\hat{n}\times\iint_{S_{PEC}} \vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds' - \frac{\hat{n}\times}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{PEC}} \vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$
- We can perform an inner-scalar product of this equation with some known vector that is tangent to S:
  - o $\left.\left\langle \vec{T}, \vec{E}^i_{\tan}\right\rangle\right|_{S_{PEC}} = -\left.\left\langle \vec{T}, \vec{E}^s_{\tan}\right\rangle\right|_{S_{PEC}}$
- Finally, this is cast in the form of a linear operator $f = \mathfrak{L}x$, where
  - ▪ $f = \left.\left\langle \vec{T}, \vec{E}^i_{\tan}\right\rangle\right|_{S_{PEC}}$, $x = \vec{J}_s$, and $\mathfrak{L}x = -\left.\left\langle \vec{T}, \vec{E}^s_{\tan}\right\rangle\right|_{S_{PEC}}$

Method of Weighted Residuals

- Next expand *x* as a set of *known* functions ϕ_n weighted by *unknown* constant coefficients α_n
  - $x \approx \sum_{n=1}^{N} \alpha_n \phi_n$
  - ϕ_n spans a function space of linearly independent functions that have support on *S*, and interpolate *x* to some polynomial order *p*.

- The expansion of *x* is applied to the integral operator:
  - $f \approx \mathfrak{L} \sum_{n=1}^{N} \alpha_n \phi_n = \sum_{n=1}^{N} \alpha_n \left(\mathfrak{L} \phi_n \right)$
  - We have *N* unknown coefficients. Presently, the above equation appears to provide only a single constraint. However, it is a continuous function over *S*.
- We need a total of *N* constraints to solve for the *N* unknown coefficients α_n.

- Introduce another set of functions φ_n (N-functions)
  - φ_n also span a function space of linearly independent functions that have support on S, and are complete to some polynomial order.
  - These functions are referred to as *testing functions*

- Define the inner product:
  - $\langle f, g \rangle = \iint_{\Omega} f(\mathbf{r}) g(\mathbf{r}) d\Omega$
  - Where Ω is the range of f and g

- Perform the inner product of $f = \mathfrak{L}x$ with each test function:
  - $\langle \varphi_m, f \rangle = \sum_{n=1}^{N} \alpha_n \langle \varphi_m, \mathfrak{L}\phi_n \rangle, \quad m = 1..N$

$$\left\langle \varphi_m, f \right\rangle = \sum_{n=1}^{N} \alpha_n \left\langle \varphi_m, \mathfrak{L}\phi_n \right\rangle, \quad m = 1..N$$

- This leads to a linear system of equations (Note, φ_m and ϕ_n are known!):
  - $\overline{f} = \overline{\overline{Z}}\,\overline{\alpha}$,
    - where, $\overline{f}$ is a column vector of length N, with m-th element: $f_m = \left\langle \varphi_m, f \right\rangle$
    - $\overline{\overline{Z}}$ is an $N \times N$ matrix, with the m,n-th element: $Z_{m,n} = \left\langle \varphi_m, \mathfrak{L}\phi_n \right\rangle$
    - $\overline{\alpha}$ is the column vector of length N of unknown coefficients α_n
- Solution:
  - $\overline{\alpha} = \overline{\overline{Z}}^{-1}\overline{f}$
  - $x \approx \sum_{n=1}^{N} \alpha_n \phi_n$
- Additional theory is needed to prove that x indeed converges to the solution of $f = \mathfrak{L}x$ in the limit $N \to \infty$, or, more practically, for a sufficiently large N.
  - We will tackle this as we move forward.