Introduction to Stiffness Analysis

The stiffness method of analysis is the basis of all commercial structural analysis programs. The focus of this chapter will be development of these equations that only take into account bending deformations, i.e., ignore axial member deformations. Within the assumptions, the stiffness method for beam and frame structures is “exact”.

In the *stiffness method of analysis*, we write *equilibrium equations* in terms of unknown *joint (node) displacements*. The number of unknowns in the stiffness method of analysis is known as the ***degree of kinematic indeterminacy***, which refers to the number of node/joint displacements that are unknown and are needed to describe the displaced shape of the structure.

One **major advantage** of the stiffness method of analysis is that the **kinematic degrees of freedom are well-defined**.

Definitions and Terminology

Positive Sign Convention: Counterclockwise moments and rotations along with transverse forces and displacements in the positive y-axis direction.

Fixed-End Forces: Forces at the “fixed” supports of the kinematically restrained structure.

Member-End Forces: Calculated forces at the end of each element/member resulting from the applied loading and deformation of the structure.

Stiffness Analysis Procedure

The steps to be followed in performing a stiffness analysis can be summarized as:

1. Determine the needed displacement unknowns at the nodes/joints and label them d_1, d_2, …, d_n in sequence where n = the number of displacement unknowns or degrees of freedom.
2. Modify the structure such that it is **kinematically determinate** or **restrained**, i.e., the identified displacements in step 1 all equal zero.
3. Calculate the member fixed-end forces in this kinematically restrained state at the nodes/joints of the restrained structure due to the member applied loads. Tables of member-end forces due to member loads for the kinematically restrained members are available later in these notes. The member-end forces are vectorially added at the nodes/joints to produce the equivalent fixed-end structure forces, which are labeled P_{fi} for i = 1, 2, …, n later in the notes.
4. Introduce a unit displacement at each displacement degree of freedom identified in step 1 one at a time with all others equal to zero and without any loading on the

structure, i.e., $d_i = 1$ with $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n = 0$ for $i = 1, 2, \ldots, n$. Sketch the displaced structure for each of these cases. Determine the member-end forces introduced as result of each unit displacement for the kinematically restrained structure. These member-end forces define the member-end stiffness coefficients, i.e., forces per unit displacement. The member-end stiffness coefficients are vectorially added at the nodes/joints to produce the structure stiffness coefficients, which are labeled S_{ij} for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

5. Eliminate the error introduced in step 3 to permit the displacement at the nodes/joints. This is accomplished by applying the negative of the forces calculated in step 3 and defines the **kinematically released structure**.
6. Calculate the unknown node/ joint displacements.
7. Calculate the member-end forces.

ILLUSTRATION

To illustrate the stiffness method of analysis, we will first consider continuous beam structures. Start off by considering the two-span beam shown in Figure 1.

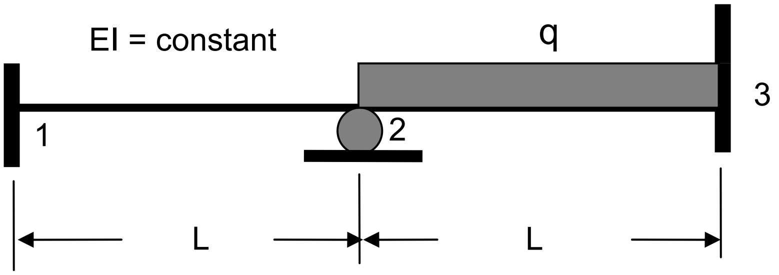


Figure 1 – Two-Span Continuous Beam

1: Determine the degree of kinematic indeterminacy.

The only unknown node/joint displacement occurs at node 2 and it is a rotational displacement. Thus, the rotation at node 2 is labeled d_1.

2: Kinematically restrain the structure such that the displacements identified in step 1 equal zero.

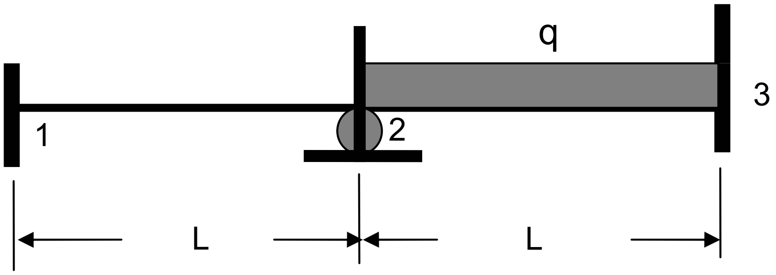


Figure 2 – Kinematically Restrained Two-Span Beam of Figure 1

The heavy vertical line drawn through the horizontal roller support at 2 signifies that node 2 is “fixed” against displacement. Thus, the rotational displacement $d_1 = 0$ for the kinematically restrained structure of Figure 2.

3: Calculate the element/member fixed-end forces for the kinematically restrained structure and vectorially add to obtain the fixed-end forces for the structure.

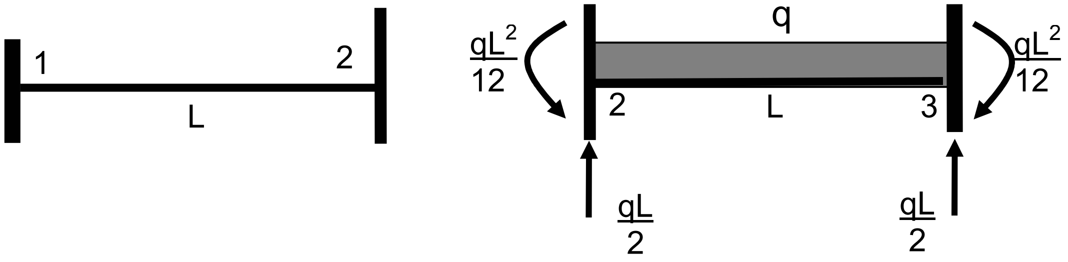


Figure 3 – “Fixed-End” Forces for the Kinematically Restrained Two-Span Beam of Figure 1

Since span element (member) 1-2 is not loaded, it will not produce any fixed-end forces. However, element (member) 2-3 is loaded and the “fixed-end” forces are labeled in Figure 3. They are simply the support reactions for the “fixed-fixed” beam.

Calculate the fixed-end forces for the structure by vectorially adding the member-end fixed-end forces.

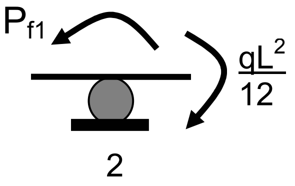


Figure 4 – Joint Equilibrium at the Kinematic Degree of Freedom for the Restrained Two-Span Beam of Figure 1

Figure 4 shows that

$$\sum_{2} M = 0 \quad \Rightarrow P_{f1} = \frac{qL^2}{12}$$

P_{f1} is drawn counterclockwise in Figure 4 since our sign convention is counterclockwise moments are positive.

4: Impose a unit displacement at each kinematic degree of freedom (DOF) to establish the structure stiffness equations.

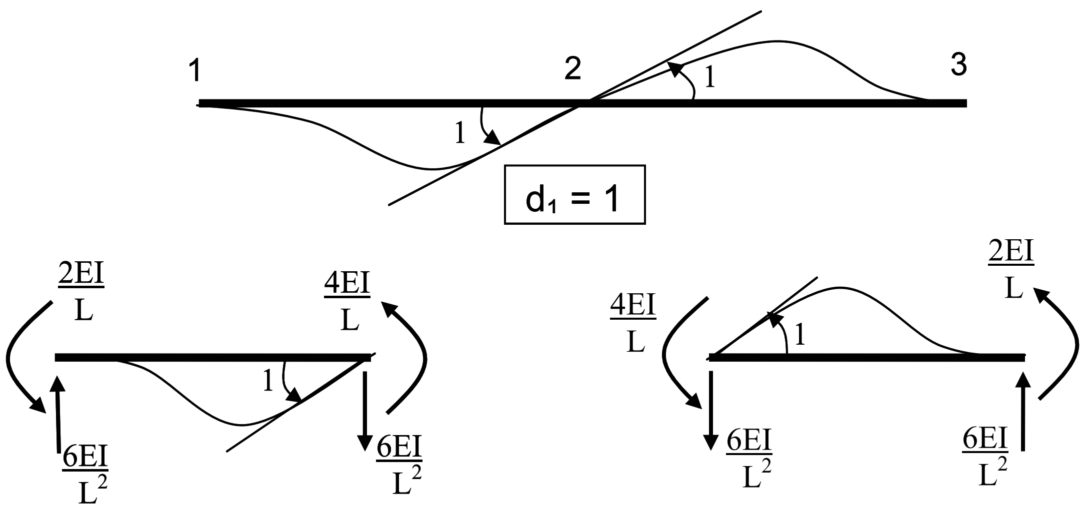


Figure 5 – Kinematically Restrained Two-Span Beam of Figure 1 Subjected to a Unit Displacement $d_1 = 1$

Figure 5 shows the displaced shape of the two-span beam for $d_1 = 1$ as well as the displaced shapes and member-end stiffness coefficients for the two elements comprising the two-span beam of Figure 1. Member-end stiffness coefficients are defined as the member-end forces resulting from the imposition of the single unit displacement for the structure as shown in Figure 5. Derivation of the member-end stiffness coefficients (forces) shown in Figure 5 and others will be covered later in the notes.

The structure stiffness equations are expressed as

$$[S]\,\{d\} = \{P\} - \{P_f\}$$

where [S] is the structure stiffness matrix; {d} is the structure displacement vector; {P} is the applied structure concentrated force vector; and $\{P_f\}$ is the structure fixed-end force vector calculated in step 3. The applied structure concentrated force vector {P} lists the point forces for each structure displacement DOF. It contains nonzero entries only at the displacement DOF where a point force or moment is applied at the corresponding displacement DOF.

The structure stiffness matrix coefficients are obtained by performing equilibrium at the nodes for each structure DOF using the member-end stiffness coefficients. These structure stiffness matrix coefficients are designated as S_{ij} and i = 1, 2, …, n and j = 1, 2, …, n.

S_{ij} ≡ force at displacement DOF i due to a unit displacement at DOF j (i.e., $d_j = 1$) with all other displacement DOF equal to zero (i.e., $d_i = 0$ for i = 1, …, j-1, j+1, …, n). Stiffness coefficients have units of force/displacement (or moment/rotation).

The structure stiffness coefficients are obtained by performing equilibrium calculations at the structure displacement degrees of freedom.

For example structure:

$$\{d\} = \{d_1\} = d_1 \Rightarrow \text{unknown}$$

$$\{P\} = \{0\} = 0$$

$$\{P_f\} = \{P_{f1}\} = qL^2/12$$

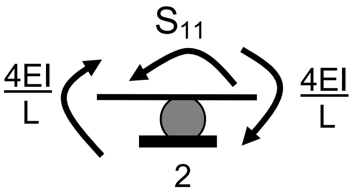


Figure 6 – Equilibrium at Kinematic DOF 1 for the Two-Span Beam of Figure 1

Performing node equilibrium at displacement DOF 1 gives (see Figure 6) gives

$$S_{11} = (4EI/L)^{12} + (4EI/L)^{23} = 8EI/L$$

5: Eliminate the error introduced in the kinematically restrained structure:

$$[S]\ \{d\} = \{P\} - \{P_f\}$$

6: Calculate the unknown structure displacements

$$\{d\} = [S]^{-1}\ (\{P\} - \{P_f\})$$

For the example structure:

$$d_1 = L/8EI\ (-qL^2/12) = -qL^3/96EI$$

7: Calculate the member-end forces.

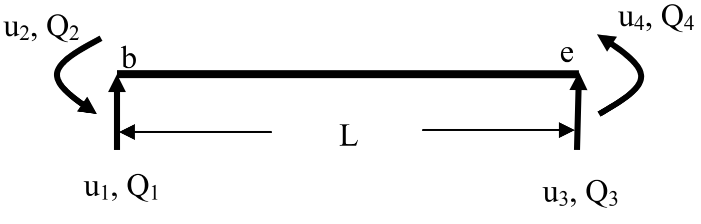


Figure 7 – Beam Element Force and Displacement Degrees of Freedom

b ≡ beginning joint/node

e ≡ end joint/node

Calculation of the member-end forces requires knowledge of the element stiffness equations. The beam member stiffness equations for the beam element of Figure 7 can be written as

$$\{Q\}_{4x1} = [k]_{4x4}\{u\}_{4x1} + \{Q_f\}_{4x1} \tag{1}$$

$$\{Q\}_{4x1} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}; \quad \{u\}_{4x1} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}; \quad \{Q_f\}_{4x1} = \begin{Bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{Bmatrix}$$

$\{Q\}$ = member-end force vector;

$\{Q_f\}$ = member fixed-end force vector;

$\{u\}$ = member-end displacement vector; and

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -\frac{12}{L^3} & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

= member stiffness matrix.

The element/member fixed-end forces are defined as $\{Q_f\} = \{Q\}$ for the kinematically determinate/restrained beam member subjected to member loading.

The member-end stiffness coefficients k_{ij} (= i, j member stiffness coefficient for i = 1, 2, 3, 4 and j = 1, 2, 3,4) is defined as:

> *The i,j stiffness coefficient is the force at kinematic degree of freedom i due to a unit displacement at kinematic degree of freedom j with all other displacements equal to zero.*

This statement on the stiffness coefficients can be expressed mathematically as

$$k_{ij} = Q_i\big|_{u_j=1} \text{ with all other } u_k = 0 \text{ for } k \neq j.$$

Equation (1) can be expanded to

$$Q_1 = V_b = \frac{12EI}{L^3}(u_1 - u_3) + \frac{6EI}{L^2}(u_2 + u_4) + Q_{f1}$$

(internal shear at beginning node b of the element)

$$Q_2 = M_b = \frac{6EI}{L^2}(u_1 - u_3) + \frac{2EI}{L}(2u_2 + u_4) + Q_{f2}$$

(internal bending moment at beginning node b)

$$Q_3 = V_e = -\frac{12EI}{L^3}(u_1 - u_3) - \frac{6EI}{L^2}(u_2 + u_4) + Q_{f3}$$

(internal shear at end node e of the element)

$$Q_4 = M_e = \frac{6EI}{L^2}(u_1 - u_3) + \frac{2EI}{L}(u_2 + 2u_4) + Q_{f4}$$

(internal bending moment at end node e)

In order to apply equation (1) to the calculation of the element end-forces, compatibility between the element displacements u_i (i=1, 2, 3, 4) and the structure displacements d_j (j = 1, 2, …, n) must be established.

Compatibility relationships for the example beam:

$u_4^{12} = d_1$; $u_2^{23} = d_1$; and all others = 0

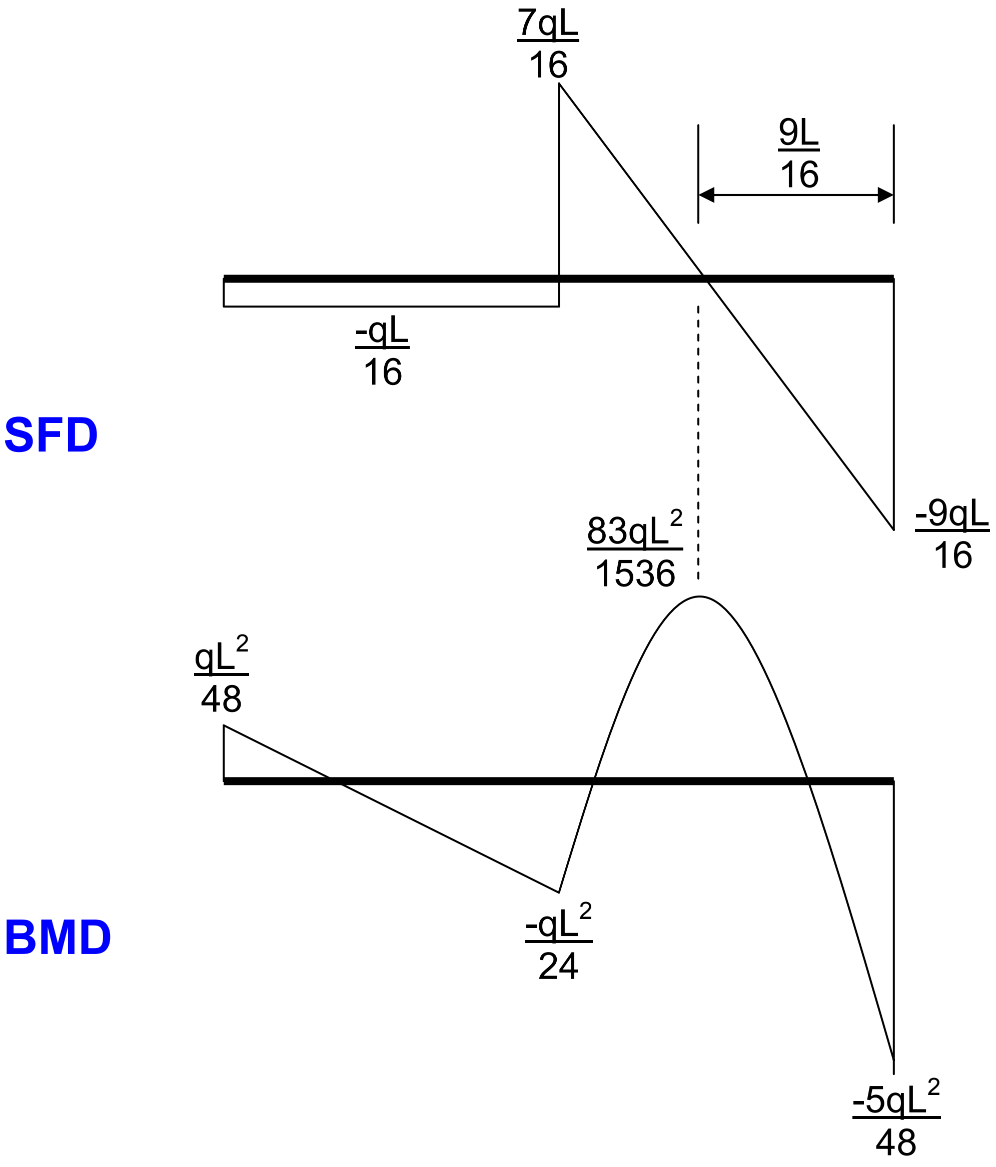
$\{Q_f\}^{12} = \{0\}$; $\{Q_f\}^{23} = qL/2 <1 \; L/6 \; 1 \; -L/6>^T$

Since only one displacement is nonzero for each member, the member end forces are

$$\{Q\}^{12} = \begin{Bmatrix} V_b \\ M_b \\ V_e \\ M_e \end{Bmatrix}^{12} = \begin{bmatrix} \frac{6EI}{L^2} \\ \frac{2EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{4EI}{L} \end{bmatrix} \left\{ \frac{-qL^3}{96EI} \right\} = \begin{Bmatrix} -\frac{qL}{16} \\ -\frac{qL^2}{48} \\ \frac{qL}{16} \\ -\frac{qL^2}{24} \end{Bmatrix}$$

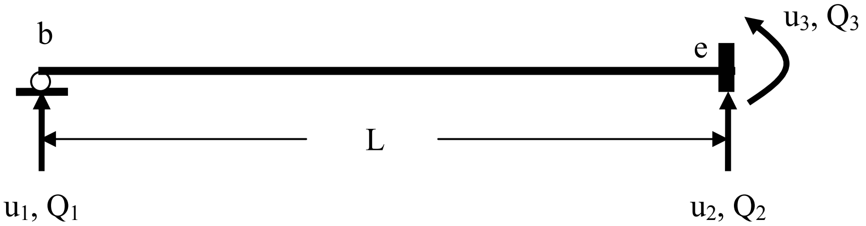
$$\{Q\}^{23} = \begin{Bmatrix} V_b \\ M_b \\ V_e \\ M_e \end{Bmatrix}^{23} = \begin{bmatrix} \frac{6EI}{L^2} \\ \frac{4EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{2EI}{L} \end{bmatrix} \left\{ \frac{-qL^3}{96EI} \right\} + \frac{qL}{2} \begin{Bmatrix} 1 \\ L/6 \\ 1 \\ -L/6 \end{Bmatrix} = \begin{Bmatrix} \frac{21qL}{48} \\ \frac{qL^2}{24} \\ \frac{27qL}{48} \\ -\frac{5qL^2}{48} \end{Bmatrix}$$

Typically with structural analysis problems, it is desired to draw the shear force and bending moment diagrams, which for the example structure is given below.


7qL/16
9L/16
-qL/16
SFD
-9qL/16
83qL^2/1536
qL^2/48
BMD
-qL^2/24
-5qL^2/48

Modified Stiffness Analysis – Zero End Moment

A simplification of the stiffness analysis is possible if we take into account the fact that the bending moment at an end simple support is zero. This leads to a reduction of one rotational degree of freedom at each end support location in which the bending moment is zero. Inclusion of this modification results in a reduction of the number of member displacement and force degrees of freedom from 4 to 3. Commensurate with this reduction is a change in the member stiffness coefficients as well as the member fixed-end forces. These changes are given below.

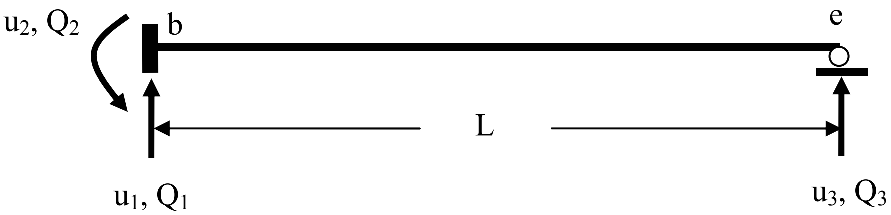


(a) Simple-Fixed Beam ($M_b = 0$)

$$\theta_b = -\frac{1}{2L}[3(u_1 - u_2) + Lu_3] - \frac{LM_{fb}}{4EI}$$

θ_b = rotation at node b

M_{fb} = fixed-end moment at node b for a "fixed-fixed" beam



(b) Fixed-Simple Beam ($M_e = 0$)

$$\theta_e = -\frac{1}{2L}[3(u_1 - u_3) + Lu_2] - \frac{LM_{fe}}{4EI}$$

θ_e = rotation at node e

M_{fe} = fixed-end moment at node e for a "fixed-fixed" beam

Modified Stiffness Matrices

For the propped cantilever beam of (a – Simple-Fixed Beam):

$$[k_M] = \frac{3EI}{L^3}\begin{bmatrix} 1 & -1 & L \\ -1 & 1 & -L \\ L & -L & L^2 \end{bmatrix} = \text{modified member stiffness matrix}$$

For the propped cantilever beam of (b – Fixed-Simple Beam):

$$[k_M] = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix} = \text{modified member stiffness matrix}$$

Modified Member-End Force Calculations

$$\{Q_M\}_{3x1} = [k_M]_{3x3}\,\{u\}_{3x1} + \{Q_{Mf}\}_{3x1}$$

$\{Q_M\} = < Q_1\ Q_2\ Q_3 >^T$ = modified member-end force vector

$\{Q_{Mf}\} = < Q_{f1}\ Q_{f2}\ Q_{f3} >^T$ = modified member fixed-end force vector

Simple-Fixed Beam

$$Q_1 = V_b = \frac{3EI}{L^3}(u_1 - u_2) + \frac{3EI}{L^2}u_3 + Q_{f1}$$

$$Q_2 = V_e = -\frac{3EI}{L^3}(u_1 - u_2) - \frac{3EI}{L^2}u_3 + Q_{f2}$$

$$Q_3 = M_e = \frac{3EI}{L^2}(u_1 - u_2) + \frac{3EI}{L}u_3 + Q_{f3}$$

Fixed-Simple Beam

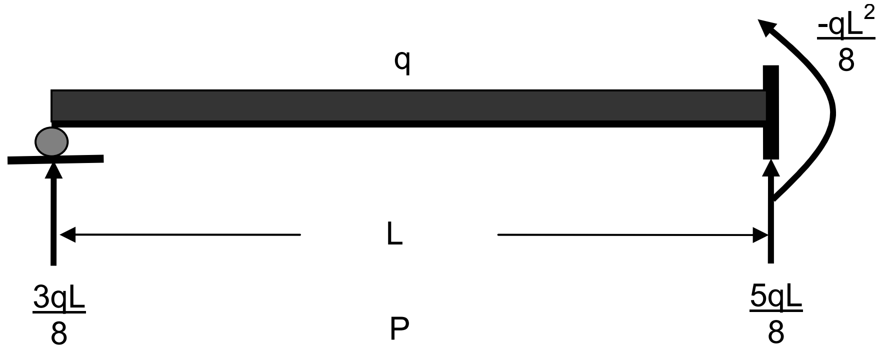
$$Q_1 = V_b = \frac{3EI}{L^3}(u_1 - u_3) + \frac{3EI}{L^2}u_2 + Q_{f1}$$

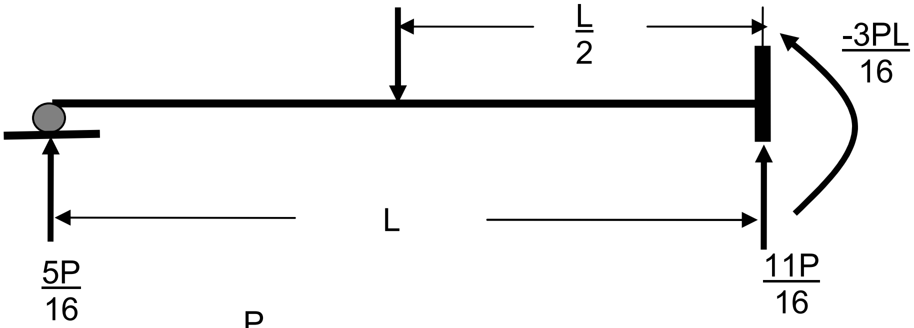
$$Q_2 = M_b = \frac{3EI}{L^2}(u_1 - u_3) + \frac{3EI}{L}u_2 + Q_{f2}$$

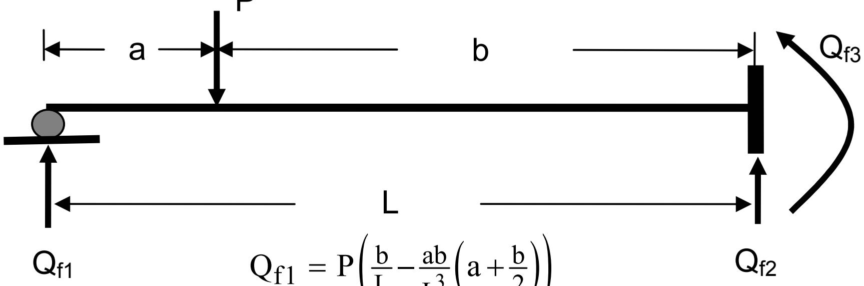
$$Q_3 = V_e = -\frac{3EI}{L^3}(u_1 - u_3) - \frac{3EI}{L^2}u_2 + Q_{f3}$$

Modified Fixed-End Force Vectors

Simple-Fixed Beams



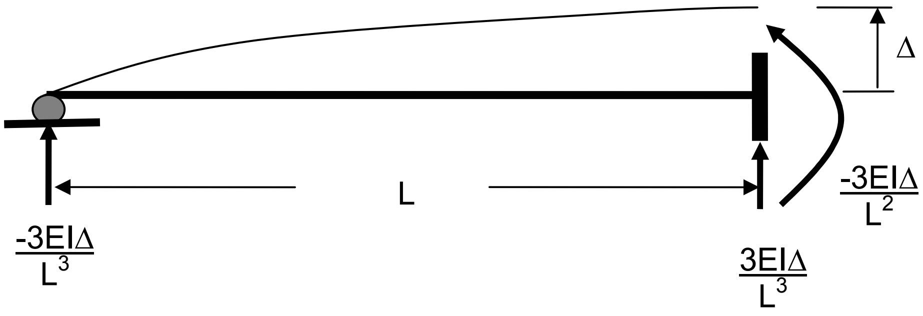




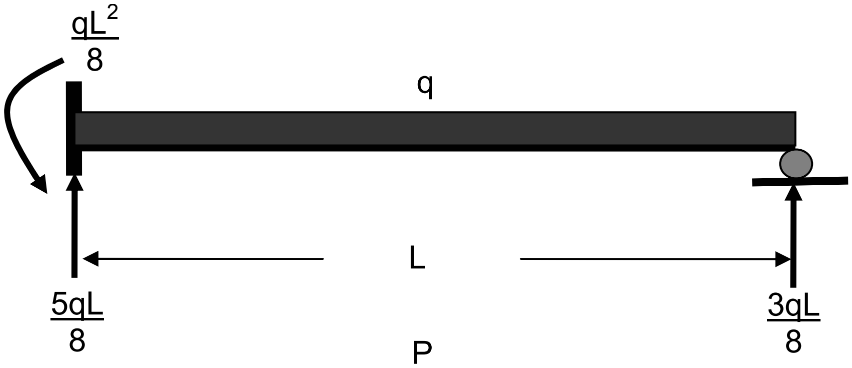
$$Q_{f1} = P\left(\frac{b}{L} - \frac{ab}{L^3}\left(a + \frac{b}{2}\right)\right)$$

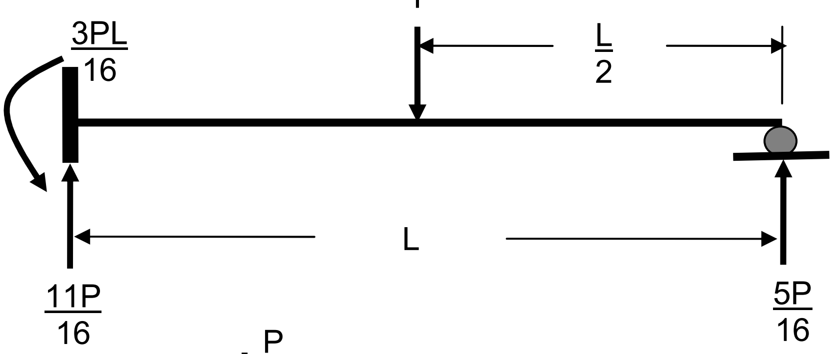
$$Q_{f2} = P\left(\frac{a}{L} + \frac{ab}{L^3}\left(a + \frac{b}{2}\right)\right)$$

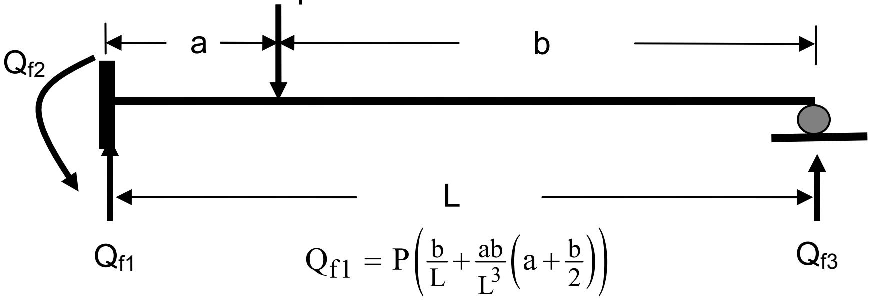
$$Q_{f3} = -P\,\frac{ab}{L^2}\left(a + \frac{b}{2}\right)$$



Fixed-Simple Beams



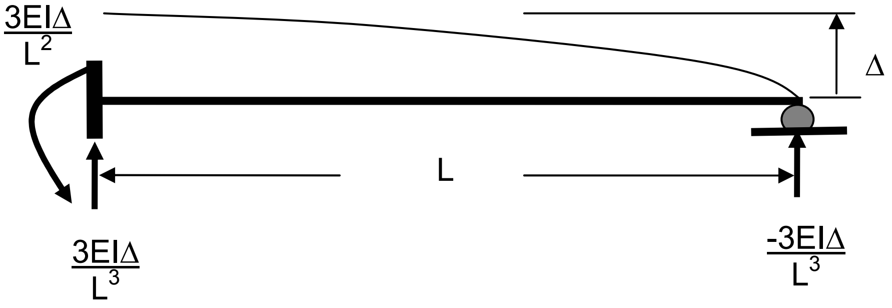




$$Q_{f1} = P\left(\frac{b}{L} + \frac{ab}{L^3}\left(a + \frac{b}{2}\right)\right)$$

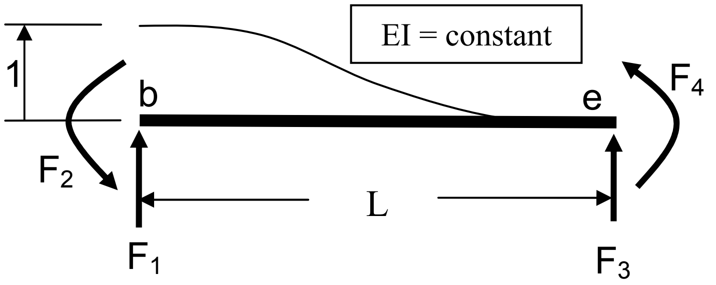
$$Q_{f2} = P\frac{ab}{L^2}\left(a + \frac{b}{2}\right)$$

$$Q_{f3} = P\left(\frac{a}{L} - \frac{ab}{L^3}\left(a + \frac{b}{2}\right)\right)$$



Stiffness Coefficient Summary

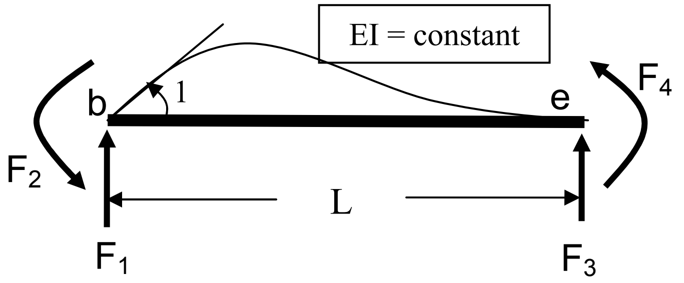
In stiffness analysis of structures that ignore axial deformation, there are only four sets of member – displacement stiffness results that need to be applied. These four are summarized in the figures below.



$$F_1 = -F_3 = \frac{12EI}{L^3}$$

$$F_2 = F_4 = \frac{6EI}{L^2}$$

(a) “Fixed-Fixed” Beam Subjected to a Unit Translational Displacement

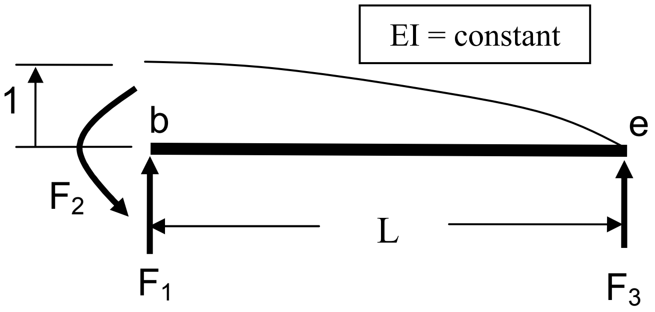


$$F_1 = -F_3 = \frac{6EI}{L^2}$$

$$F_2 = 2F_4 = \frac{4EI}{L}$$

(b) “Fixed-Fixed” Beam Subjected to a Unit Rotational Displacement

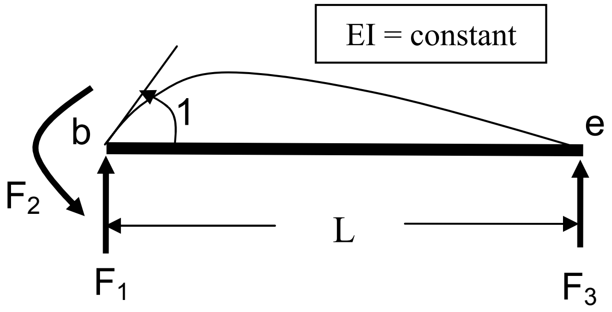
Calculations for displacements at end e are similar to those shown for end b. Figures (a) and (b) simply show the forces required to cause the unit displacement and the corresponding reaction forces at the zero displacement degrees of freedom. Compare your displaced drawing with the forces shown. Force magnitudes will be the same. Whether the force is positive or negative depends on whether it is in the same direction as the corresponding displacement on the structure (positive) or in the opposite direction (negative).



$$F_1 = -F_3 = \frac{3EI}{L^3}$$

$$F_2 = \frac{3EI}{L^2}$$

(c) “Simple-Fixed” Beam Subjected to a Unit Translational Displacement



$$F_1 = -F_3 = \frac{3EI}{L^2}$$

$$F_2 = \frac{3EI}{L}$$

(d) “Simple-Fixed” Beam Subjected to a Unit Rotational Displacement

Calculations for displacements at end e are similar to those shown for end b. Figures (c) and (d) simply show the forces required to cause the unit displacement and the corresponding reaction forces at the zero displacement degrees of freedom. Compare your displaced drawing with the forces shown. Force magnitudes will be the same. Whether the force is positive or negative depends on whether it is in the same direction as the corresponding displacement on the structure (positive) or in the opposite direction (negative).

These results can be applied directly to frames.