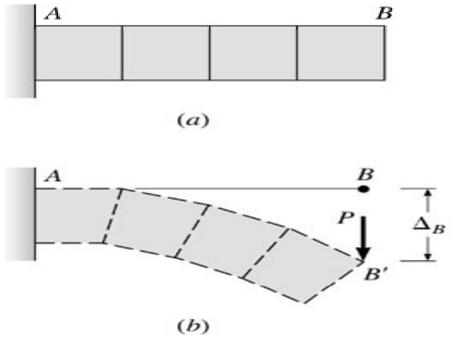
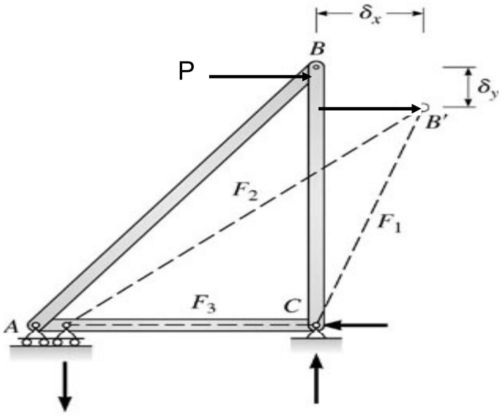
# Structural Displacements



Beam Displacement

# Structural Displacements



Truss Displacements

The deflections of civil engineering structures under the action of usual design loads are known to be small in relation to both the overall dimensions and member lengths. "Why bother to compute deflections?" Basically, the design engineer must establish that the predicted design loads will not result in large deflections that may lead to structural failure, impede serviceability, or result in an aesthetically displeasing and distorted structure.

Several examples which demonstrate the value of deflection analysis include (Tartaglione, 1991):

1. Wind forces on tall buildings have been known to produce excessive lateral deflections that have resulted in cracked windows and walls, as well as discomfort to the occupants.

2. Large floor deflections in a building are aesthetically unattractive, do not inspire confidence, may crack brittle finishes or cause other damage, and can be unsafe.

3. Floor systems are often designed to support motor-driven machines or sensitive equipment that will run satisfactorily only if the support system undergoes limited deflections.

4. Large deflections on a railway or highway structural support system may impair ride quality, cause passenger discomfort, and be unsafe.

5. Deflection control and camber behavior of pre-stressed concrete beams during various stages of construction and loading are vital for a successful design.

6. Deflection computations serve to establish the vibration and dynamic characteristics of structures that must withstand moving loads, vibration, and shock environment -- inclusive of seismic design loads.

**Elastic Deformations** ≡ structure deflections disappear and the structure regains its original shape when the actions causing the deformations are removed.

**Permanent deformations** of structures are referred to as inelastic or plastic deformations.

This course will focus on **linear elastic deformations**. Such deformations vary linearly with applied loads and the principle of superposition is valid for such structures. Furthermore, since the deflections are expected to be small, deflections are measured with respect to the original, undeformed or reference geometry.

## Work-Energy Methods

Work-energy methods for truss, beam and frame structures are considered. Such methods are based on the principle of ***conservation of energy***, which states that the work done by a system of forces applied to a structure (**W**) equals the strain energy stored (**U**) in the structure.

This statement is based on slowly applied loads that do not produce kinetic energy, which can be written as

$$W = U$$

**A disadvantage of work-energy methods is that only one displacement component or rotation can be computed with each application**.

**Work** ≡ force (moment) times displacement (rotation) in the force (moment) direction

Differential work of Fig. 1 can be expressed as

$$dW = P\,(d\Delta)$$

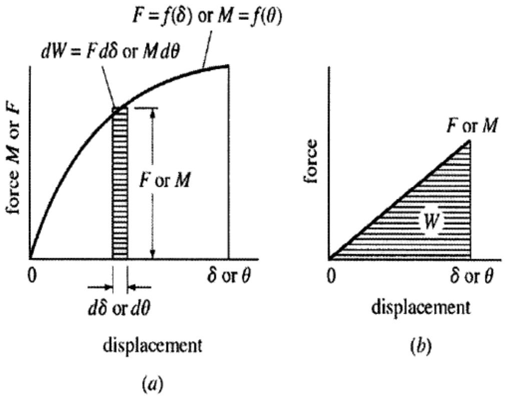


Figure 1. Force versus Displacement Curves

For P = F (force), Δ equals displacement δ:

$$W = \int_0^{\delta} F\,d\delta \qquad (1)$$

For P = M (moment), Δ equals rotation θ:

$$W = \int_0^{\theta} M\,d\theta \qquad (2)$$

Eqs. (1, 2) indicate that work is simply the area under the force – displacement (or moment – rotation) diagrams shown in Fig. 1.

## Linear Elastic Structure

$$W = \frac{1}{2} F\,\delta$$

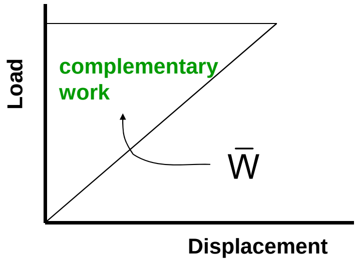
$$W = \frac{1}{2} M\,\theta$$

## Complementary Work

The area above the load-displacement diagrams of Fig.1 is known as **complementary work**, $\overline{W}$ as shown in Fig 2. For a linear-elastic system:

$$W = \overline{W} = \frac{1}{2} P\,\Delta$$

Fig. 2. Complementary Work



Direct use of work-energy calculations is only capable of calculating displacements at the location of an applied point force and rotations at the point of application of a point couple; obviously a very restrictive condition. Consequently, virtual work principles are developed in the subsequent sections.

## Virtual Work

Virtual (***virtual*** ≡ *imaginary, not real, or in essence but not in fact*) work procedures can produce a single displacement component at any desired location on the structure. To calculate the desired displacement, a *dummy or virtual load* (normally of unit magnitude) is applied at the location and in the direction of the desired displacement component. Forces associated with this virtual force are subscripted with a V.

Use of a virtual force in calculating virtual work is defined as the principle of virtual forces (which will be the focus of this chapter):

**Principle of Virtual Forces**

*If a deformable structure is in equilibrium under a virtual system of forces, then the external work done by the virtual forces going through the real displacements equals the internal virtual work done by the virtual stress resultants going through the real displacement differentials.*

Alternatively, if virtual displacements are applied then the virtual work is defined as the principle of virtual displacements:

**Principle of Virtual Displacements**

*If a deformable structure is in equilibrium and remains in equilibrium while it is subject to a virtual distortion, the external virtual work done by the external forces acting on the structure is equal to the internal virtual work done by the stress resultants.*

The virtual work principles (forces and displacements) are based on conserving the change in energy due to the applied virtual load or displacement, which can be expressed mathematically as

$$\bar{W}_V = \bar{U}_V$$

for the principle of virtual forces, which is the focus of this chapter, where again the **overbar signifies complementary energy**. The real and virtual complementary external work is shown schematically in Fig. 3.

---

**$P_V$ = virtual force or moment**

**$\Delta$ = real displacement $\delta$ or rotation $\theta$**

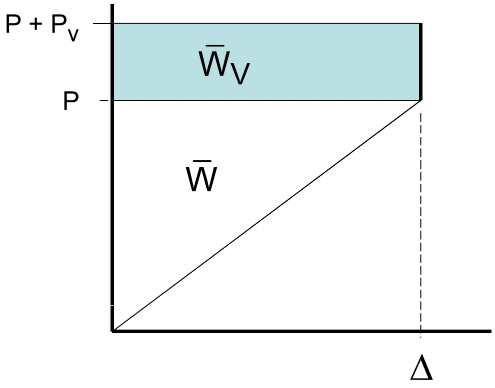


Fig. 3. Complementary Real and Virtual Works

---

## Complementary Axial Strain Energy

For a single axial force member subjected to a real force F, the complementary strain energy (internal work) is

$$\bar{U} = \frac{F}{2}\delta$$

$$\delta = \frac{FL}{EA}$$

$$\Rightarrow \bar{U} = \frac{F^2 L}{2EA}$$

---

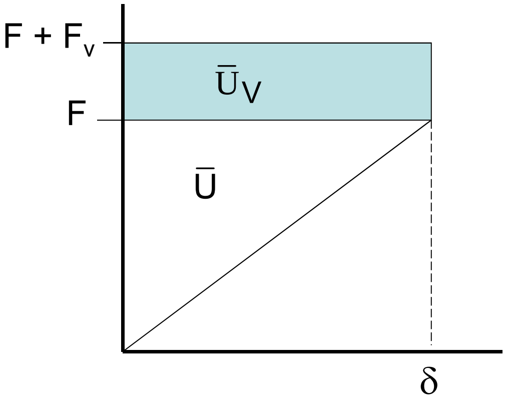
For a single axial force member in equilibrium subjected to a virtual force $F_V$, the virtual complementary strain energy (virtual complementary internal work) is

$$\bar{U}_V = F_V \delta$$

$$\Rightarrow \bar{U}_V = F_V \frac{FL}{EA}$$

Real and virtual complementary strain energies for a single member are shown schematically in Fig. 4.

Fig. 4. Complementary Real and Virtual Strain Energies for a Single Truss Member



For a truss structure:

$$\bar{U}_V = \sum_{i=1}^{m} \bar{U}_{Vi} = \sum_{i=1}^{m} F_{Vi}\,\delta_i$$

$\bar{U}_{Vi}$ = Complementary Virtual Strain Energy for Truss Member i

$\delta_i$ = Real Member i Deformation

$\delta_i = \dfrac{F_i L_i}{E_i A_i}$ ; for a mechanically loaded truss member

$\delta_i = \alpha_i L_i \Delta T_i$ ; for a thermally loaded truss member

α = linear coefficient of thermal expansion

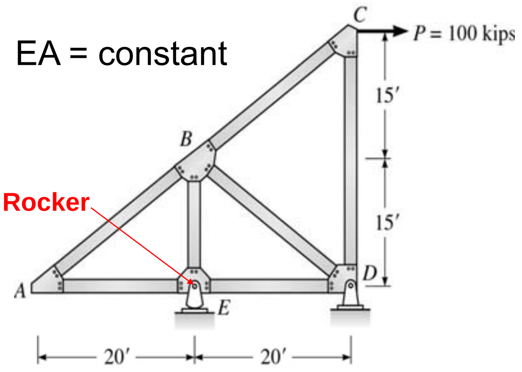
ΔT = change in temperature

$\delta_i = \Delta L_{fi}$ ; for a fabrication error of $\Delta L_f$ in the truss member

**Non-mechanical $\delta_i$ are positive if they produce a positive change in member length consistent with tension positive forces in truss members.**

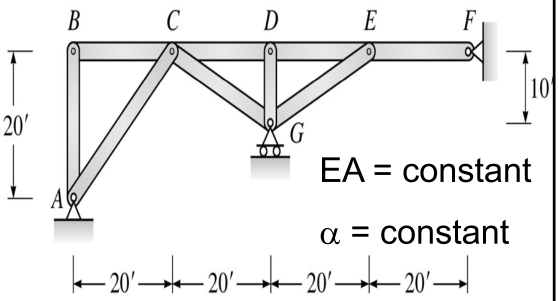
## Example Deflection Calculation – Mechanically Loaded Truss Structure

EA = constant



Calculate the horizontal displacement at C.

## Example Deflection Calculation – Thermally Loaded Truss Structure



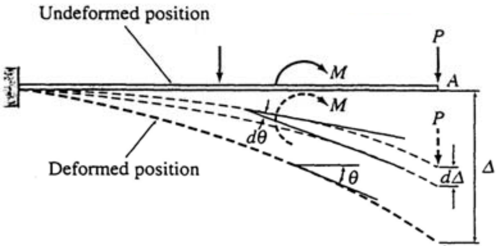
EA = constant

α = constant

Calculate the horizontal displacement at G if the top chord members are subjected to a temperature increase of ΔT = 100.

Equation of condition at C!

## Complementary Bending Strain Energy



$$d\bar{U} = \frac{M}{2} d\theta$$

$$d\theta = \frac{M}{EI} dx$$

$$\Rightarrow \bar{U} = \frac{1}{2} \int_L M \frac{M}{EI} dx$$

$$d\bar{U}_V = M_V \, d\theta$$

$$\Rightarrow \bar{U}_V = \int_L M_V \frac{M}{EI} dx$$

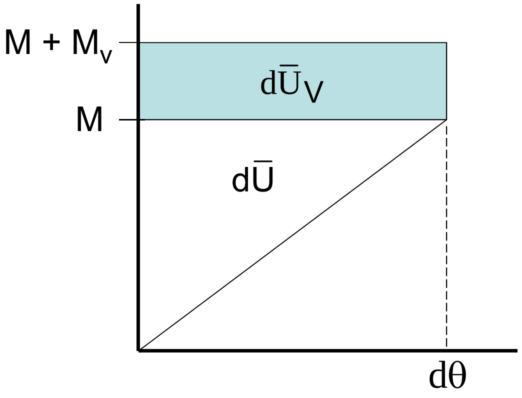


Fig. 6. Complementary Real and Virtual Strain Energies for a Differential Bending Segment

$M$ = real bending equation due to the external applied loading

$M_V$ = $M_V^\delta$ for a virtual moment equation due to a unit force for a point displacement $\delta$ calculation at the desired point in the assumed direction of the displacement

= $M_V^\theta$ for a virtual moment equation for a unit virtual couple for a point rotation $\theta$ calculation at the desired point in the assumed rotation direction

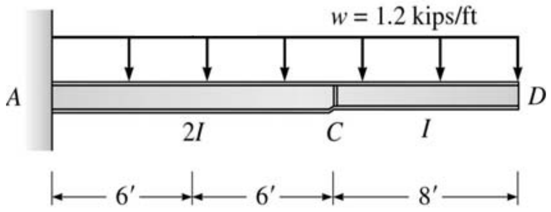
For a multi-segment beam:

$$\bar{U}_V = \sum_{i} \int_{L_i} M_{Vi} \frac{M_i}{E I_i} dx$$

where i designates beam segment.

---

## Beam Deflection Example



Calculate the tip displacement and rotation for the cantilever beam.

---

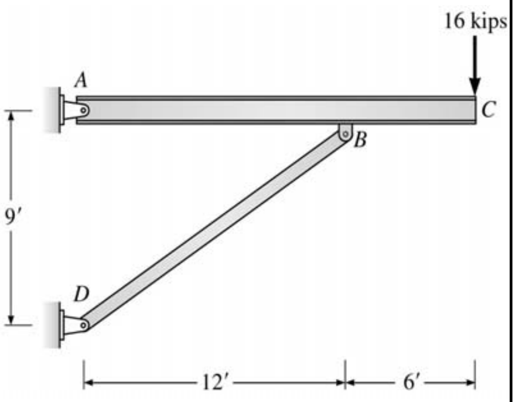
For a frame structure:

$$\bar{U}_V = \sum_{j=1}^{m} \int_{L_j} M_{Vj} \frac{M_j}{E I_j} dx$$

where m equals the number of frame members. **Note, axial deformation has been ignored**. Also, if a frame member is composed of multi-segments, then a summation over the segments must also be included.

---

## Frame Deflection Example



Calculate the vertical displacement and rotation at C.