Indeterminate Analysis Force Method[1]

- The force (flexibility) method expresses the relationships between displacements and forces that exist in a structure.
- Primary objective of the force method is to determine the chosen set of excess unknown forces and/or couples – **redundants**.
- The *number of redundants* is *equal to the degree of static indeterminacy* of the structure.

[1]Also see pages 115 – 141 in your class notes.

1

Description of the Force Method Procedure

1. Determine the degree of static indeterminacy.

- Number of **releases*** equal to the degree of static indeterminacy are applied to the structure.
- **Released structure** is referred to as the ***primary structure***.
- Primary structure must be chosen such that it is geometrically stable and statically determinate.
- Redundant forces should be carefully chosen so that the primary structure is easy to analyze

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*** Details on releases are given later in these notes.**

Force Method – con't

2. Calculate "errors" (displacements) at the primary structure redundants. These displacements are calculated using the method of virtual forces.

3. Determine displacements in the primary structure due to unit values of redundants (method of virtual forces). These displacements are required at the same location and in the same direction as the displacement errors determined in step 2.

3

Force Method – con't

4. Calculate redundant forces to eliminate displacement errors.

- Use superposition equations in which the effects of the separate redundants are added to the displacements of the released structure.
- Displacement superposition results in a set of n linear equations (n = number of releases) that express the fact that there is zero relative displacement at each release.

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Force Method – con't

- These **compatibility equations** guarantee a final displaced shape consistent with known support conditions, i.e., the structure fits together at the n releases with no relative displacements.

5. Hence, we find the forces on the original indeterminate structure. They are the sum of the correction forces (redundants) and forces on the released structure.

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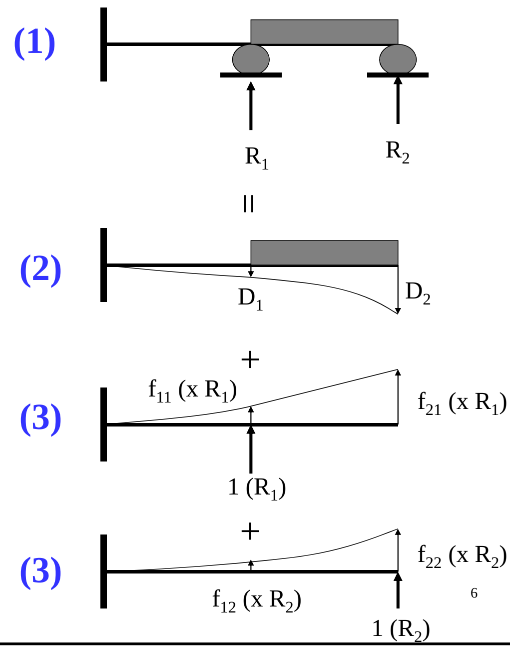
Flexibility Analysis



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$$f_{11} R_1 + f_{12} R_2 + D_1 = 0$$

(4)

$$f_{21} R_1 + f_{22} R_2 + D_2 = 0$$

Solve for R_1 and R_2.

Using matrix methods:

[F] {R} = -{D}

$\Rightarrow$ {R} = -$[F]^{-1}$ {D}

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$$[F] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$ ≡ flexibility matrix

$[F]^{-1}$ (≡ inverse flexibility matrix)

$$= \frac{1}{f_{11}f_{22} - f_{12}f_{21}} \begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix}$$

$$\{D\} = \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}$$ ≡ primary structure displacement vector

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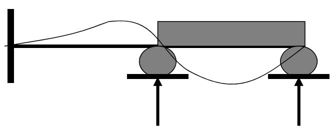
$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$ ≡ redundant force vector

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{-1}{\det[F]} \begin{Bmatrix} f_{22}D_1 - f_{12}D_2 \\ -f_{21}D_1 + f_{11}D_2 \end{Bmatrix}$$

$$\det[F] = f_{11}f_{22} - f_{12}f_{21}$$

(5)



With R_1 and R_2 known, remaining structure is statically determinate.

Releases

Release is a break in the continuity of the elastic (displacement) curve.

- One release only breaks a single type of continuity.
- Figure 1 shows several types of releases.
- Common release is the support reaction, particularly for continuous beams.

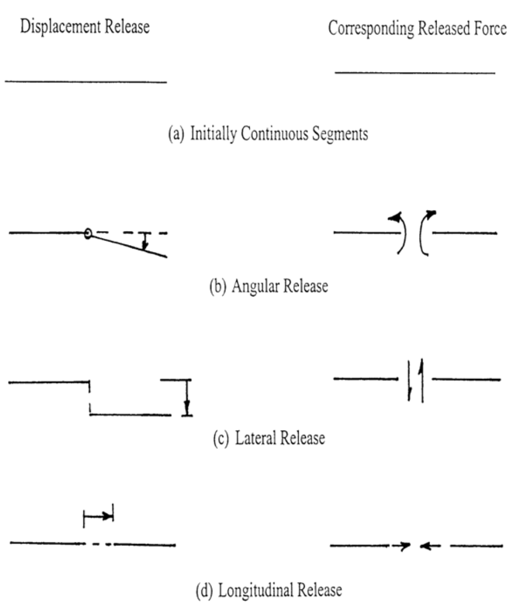


Fig. 1 – Definition of Releases

Flexibility Equations

Primary structure displacements at the releases are related to the unknown redundant forces via

$$-D_i = f_{ij} R_j \qquad (1)$$

f_{ij} ≡ displacement at release i due to a unit force in the direction of and at release j; flexibility coefficients.

Equation 1 for the case of three redundant forces is expressed as

$$-D_1 = f_{11}R_1 + f_{12}R_2 + f_{13}R_3$$
$$-D_2 = f_{21}R_1 + f_{22}R_2 + f_{23}R_3 \quad (2a)$$
$$-D_3 = f_{31}R_1 + f_{32}R_2 + f_{33}R_3$$

Matrix form of (2a)

$$-\{D\} = [F]\{R\} \qquad \textbf{(2b)}$$

$$\{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = <D_1 \;\; D_2 \;\; D_3>^T$$

= **displacement vector at the redundant degrees of freedom**

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = <R_1 \;\; R_2 \;\; R_3>^T$$

= **redundant force vector**

$$[F] = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

= **flexibility matrix**

Displacement Calculations – Method of Virtual Forces

$$D_i = \int F_{Vi}\, d\ell + \int M_{Vi}\, d\phi \qquad \textbf{(3)}$$

subscript i $\Rightarrow$ direction of R_i at release i

$d\ell$ = differential axial displacement

$d\phi$ = differential rotational displ

Flexibility Coefficients – Method of Virtual Forces

$$f_{ij} = f_{ij}^a + f_{ij}^b \qquad \textbf{(4)}$$

$$f_{ij}^a = \int F_{Vi} \frac{F_{Vj}}{EA(x)} dx$$

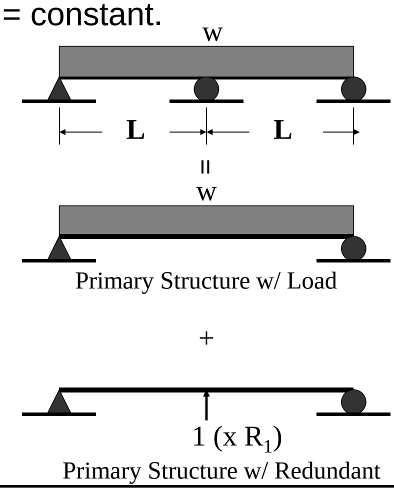
≡ axial deformation influence coefficient

$$f_{ij}^b = \int M_{Vi} \frac{M_{Vj}}{EI(x)} dx$$

≡ bending deformation influence coefficient

Force Method Examples

1. Calculate the support reactions for the two-span continuous beam, EI = constant.



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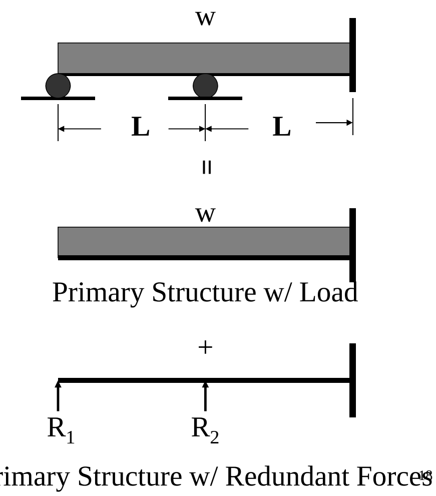
2. Calculate the support reactions for the two-span continuous beam, EI = constant.



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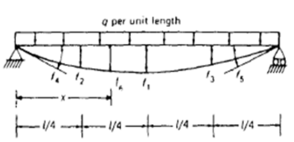
Prismatic Member Displacements

The following table gives the displacements in beams of constant flexural rigidity EI and constant torsional rigidity GJ, subjected to the loading shown on each beam. The positive directions of the displacements are downward for translation, clockwise for rotation. The deformations due to shearing forces are neglected.

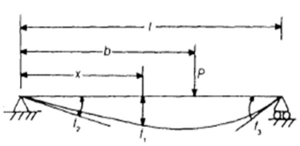


$$f_1 = \frac{5}{384}\frac{ql^4}{EI}$$

$$f_2 = f_3 = \frac{19}{2048}\frac{ql^4}{EI}$$

$$f_4 = -f_5 = \frac{ql^3}{24EI}$$

$$f_6 = \frac{qx}{24EI}(l^3 - 2lx^2 + x^3)$$

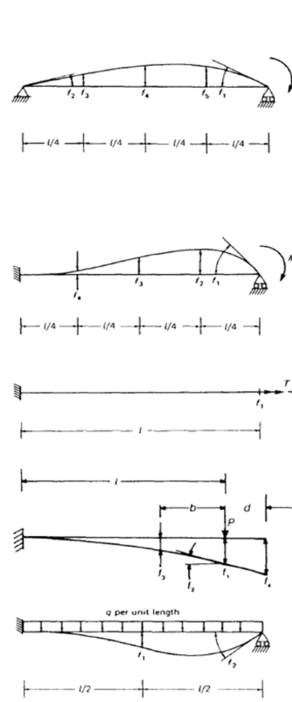


$$f_1 = \frac{P(l-b)x}{6lEI}(2lb - b^2 - x^2) \quad \text{when } x \leqslant b$$

$$f_1 = \frac{Pb(l-x)}{6lEI}(2lx - x^2 - b^2) \quad \text{when } x \geqslant b$$

$$f_2 = \frac{Pb(l-b)}{6lEI}(2l-b) \qquad f_3 = -\frac{Pb}{6lEI}(l^2 - b^2)$$

When $b = l/2$, $f_2 = -f_3 = Pl^2/(16EI)$, and $f_1 = Pl^3/48EI$ at $x = l/2$.



$$f_1 = \frac{Ml}{3EI}$$

$$f_2 = -\frac{Ml}{6EI}$$

$$f_3 = -\frac{15Ml^2}{384EI}$$

$$f_4 = -\frac{Ml^2}{16EI}$$

$$f_5 = -\frac{21Ml^2}{384EI}$$

$$f_1 = \frac{Ml}{4EI}$$

$$f_2 = -\frac{9Ml^2}{256EI}$$

$$f_3 = -\frac{Ml^2}{32EI}$$

$$f_4 = -\frac{3Ml^2}{256EI}$$

$$f_1 = \frac{Tl}{GJ}$$

(Effect of warping ignored)

$$f_1 = \frac{Pl^3}{3EI}$$

$$f_2 = Pl^2/2EI$$

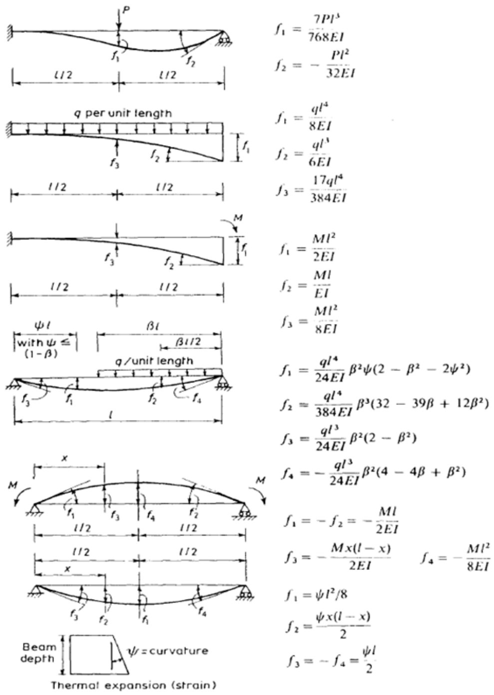
$$f_4 = f_1 + df_2$$

$$f_3 = \frac{Pl^3}{3EI}\left(1 - \frac{3b}{2l} + \frac{b^3}{2l^3}\right)$$

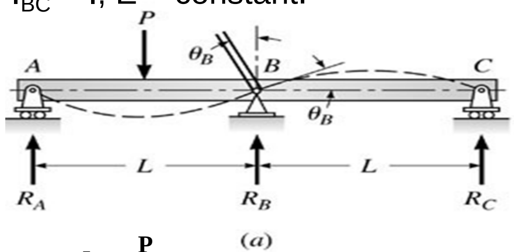
for $0 \leqslant b \leqslant l$

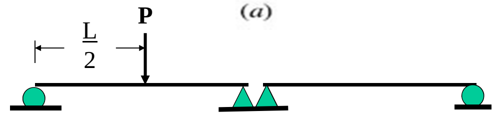
$$f_1 = \frac{ql^4}{192EI}$$

$$f_2 = -\frac{ql^3}{48EI}$$

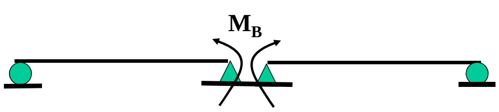


3. Calculate the support reactions for the two-span continuous beam using the internal moment at B as the redundant force, I_{AB} = 2I and I_{BC} = I; E = constant.





Primary Structure w/ Loading



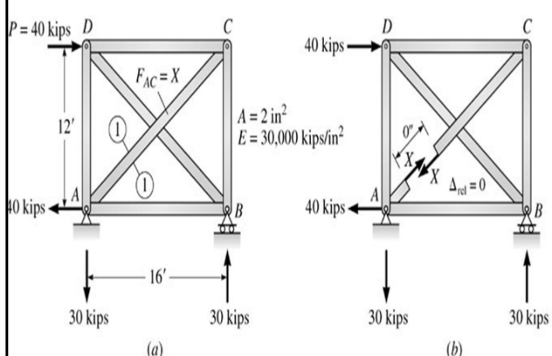
Primary Structure w/ Redundant

D_B = ____________________

f_{BB} = ____________________

M_B = ____________________

4. Calculate the bar forces for the statically indeterminate truss.



Statically Indeterminate Truss

Statically Determinate Released Truss

(Redundant X)

Truss Calculations

Mem	L (in)	F	F_V	F_VFL
AB	192"	40	-4/5	-6144
BC	144"	0	-3/5	0
CD	192"	0	-4/5	0
DA	144"	30	-3/5	-2592
AC	240"	0	1	0
BD	240"	-50	1	-12000

$$F_{AC} = -\left[\sum \frac{F_V F L}{E A}\right] / f_{AC,AC} = 20736/829.44$$

$= 25$ kips

Nonmechanical Loading

$$[F]\{R\} = -(\{D\} + \{D^{\Delta}\}) \quad \textbf{(5)}$$

$$\{D^{\Delta}\} = \langle D_1^{\Delta} \; D_2^{\Delta} \ldots D_n^{\Delta} \rangle^T$$

= relative dimensional change displacements calculated using principle of virtual forces

Displacements due to dimension changes are all relative displacements, as are all displacements corresponding to releases. They are ***positive when they are in the same vector direction as the corresponding release (redundant)***.

Structure Forces

Once the redundant forces are calculated from Eq. (5), all other support reactions and internal member forces can be calculated using static equilibrium along with the appropriate free body diagrams.

This is possible since the force method of analysis has been used to determine the redundant forces or the forces in excess of those required for static determinacy.

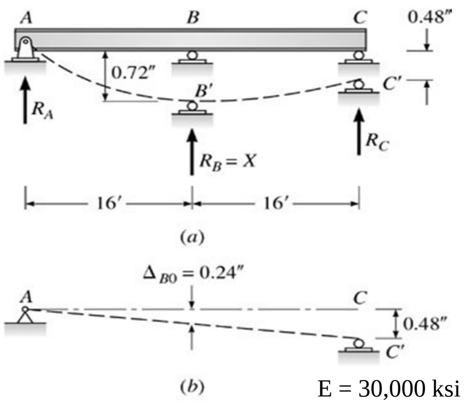
Mathematical Expressions

Calculation of the **non-redundant forces A_i** (support reactions, internal shears and moments, truss member forces) can be expressed using **superposition** as

$$A_i = A_i^p + \sum_{j=1}^{N_R} (A_{ui})_j R_j \quad \textbf{(6)}$$

where A_i^p = **desired action A_i on the primary structure due to the applied loading**; $(A_{ui})_j$ = **action A_i on the primary structure due to a unit virtual force at redundant R_j** and N_R = number of redundants.

Example Beam Problem – Nonmechanical Loading



E = 30,000 ksi

I = 288 in^4

(a) Given structure

(b) Primary structure

The interesting point of this example is that the flexibility equation will have a nonzero right hand side since the redundant displacement is prescribed to equal 0.72" downward. Thus the flexibility equation is

$$f_{BB} R_B = d_B - D_B^{\Delta} \qquad (7)$$

where

d_B = prescribed displacement at redundant B

= -0.72" since R_B is positive upward

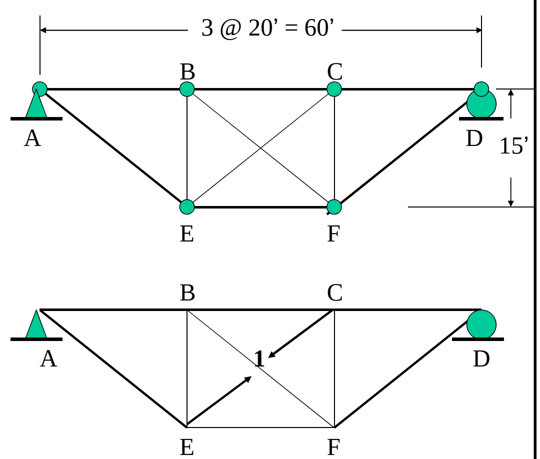
D_B^{Δ} = -0.24"

$d_B - D_B^{\Delta}$ = relative displacement at redundant B

Truss Example – Nonmechanical Loading

For the truss structure on the next page, compute the redundant bar EC member force if the temperature in bar EF is increased 50 ^oF and member BF is fabricated 0.3 in. too short. EA = constant = 60,000 kips and α = 6x10^{-6} /^{o}F.

Truss Example



Primary Structure Subjected to $F_{CE} = 1$

Truss Example Calculations

Mem	L	F_V	$F_V F_V L$
AB	240"	0	0
AE	300"	0	0
BC	240"	-4/5	153.6
BE	180"	-3/5	64.8
BF	300"	1	300
CD	240"	0	0
CE	300"	1	300
CF	180"	-3/5	64.8
DF	300"	0	0
EF	240"	-4/5	153.6

$$f_{CE,CE} = \frac{1}{EA}\sum_{i=1}^{m} F_{Vi}\; F_{Vi} L_i$$

$$D_{CE}^{\Delta} = \sum_{i=1}^{m} F_{Vi}\; \delta_i^{\Delta}$$

$$\delta_{EF}^{\Delta} = \alpha \, \Delta T_{EF} \, L_{EF} = 0.072"$$

$$\delta_{BF}^{\Delta} = \Delta \ell_{BF} = -0.3"$$

$$f_{CE,CE} F_{CE} + D_{CE}^{\Delta} = 0$$

Displacement Calculations

Displacements for the statically indeterminate structure can be calculated using the exact member deformations for a truss or exact shear and moment expressions along with the virtual force expressions on the primary structure.

For a **truss structure**, calculation of a joint displacement Δ using the principle of virtual forces results in

$$1\,(\Delta) = \sum_{i=1}^{m} F_{Vi}\, \delta_i + \Delta^{\Delta}$$

$$= \sum_{i=1}^{m} F_{Vi} \left(\frac{F_i L_i}{EA_i} + \delta_i^{int} \right) + \Delta^{\Delta} \quad \textbf{(8)}$$

F_{Vi} = primary structure member forces due to the application of a unit virtual force at the joint for which the displacement Δ is desired and in the direction of Δ

Δ^{Δ} = primary structure displacement at desired displacement due to nonmechanical effects

δ_i = exact member displacements that are obtained for the statically indeterminate structure using the calculated redundant forces to determine all the member forces within the truss structure

δ_i^{int} = member displacements due to nonmechanical loading on the member

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For a **frame structure**, in which **shear and axial deformations are ignored**, the displacements are calculated as

$$1(\Delta) = \sum_{i=1}^{m} \int_0^L M_{Vi}^{\Delta} \left(\frac{M_i}{EI_i} + \kappa_i^{int} \right) dx + \Delta^{\Delta} \quad \textbf{(9a)}$$

$$1(\theta) = \sum_{i=1}^{m} \int_0^L M_{Vi}^{\theta} \left(\frac{M_i}{EI_i} + \kappa_i^{int} \right) dx + \theta^{\Delta} \quad \textbf{(9b)}$$

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M_{Vi}^{Δ}, M_{Vi}^{θ} = primary structure virtual moments based on the desired displacement Δ or rotation θ

Δ^{Δ}, θ^{Δ} = primary structure displacements at Δ or rotation θ due to environmental loads or causes

κ_i^{int} = primary structure initial curvature strain caused by nonmechanical loading

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In **Eqs. (9a) and (9b) the moment expressions are exact based on the statically indeterminate structure subjected to the external loads with the redundant forces known from the flexibility analysis.**

Equations (8), (9a), and (9b) are correct only because exact real member forces are used in the calculation of the desired displacements.

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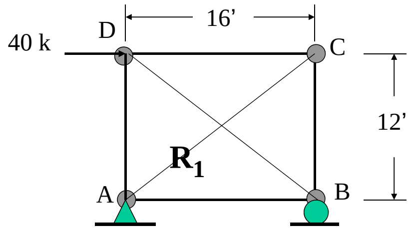
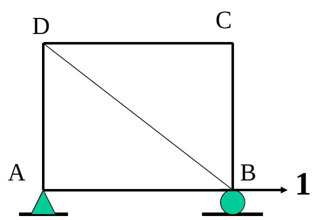
Calculate the horizontal displacement at joint B for the statically indeterminate truss.





Primary Structure Subjected to Virtual Loading

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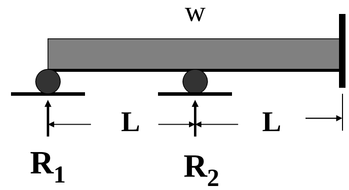
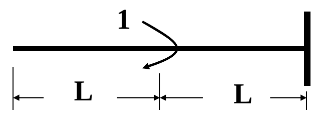
Calculate the rotation at the center support for the two-span continuous beam, EI = constant.





Primary Structure w/ Virtual Load at Desired Displacement Location

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Alternatively, you can express the desired displacement calculations also in matrix form following the usual superposition process of the force method of analysis:

$$\{\delta\} = [F_\delta]\{R\} + \{D_\delta\} + \{D_\delta^\Delta\} \quad (10)$$

where $\{\delta\}$ = vector of desired displacements; $\{D_\delta\}$, $\{D_\delta^\Delta\}$ = vector of desired displacements for the primary structure for both mechanical and non-mechanical loadings, respectively; $[F_\delta]$ =

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matrix of displacement influence coefficients at the desired displacement locations due to unit values of the redundant forces $\{R\}$. Stated mathematically, the coefficients of $[F_\delta]$ are

$$F_{\delta ij} = \delta_i\big|_{R_j=1} \quad (11)$$

which simply states that the displacement influence coefficients equal the displacement at desired displacement i on the primary structure due to a unit force at redundant j on the primary structure.

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