EE611
Deterministic Systems

Controllability and Observability Discrete Systems

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Canonical Decompositions

Given the Controllability matrix of an *n* dimensional system that is not controllable:

$$\mathrm{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} & \ldots & \mathbf{A}^{\mathbf{n}-1}\mathbf{B} \end{bmatrix} \quad \text{where } \rho(\mathrm{C}) = n_1 < n$$

Define equivalence transformation

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \ldots & \mathbf{q}_{\mathbf{n}_1} & \ldots & \mathbf{q}_\mathbf{n} \end{bmatrix} \qquad \mathbf{P} = \mathbf{Q}^{-1}$$

where the first n_1 columns of **Q** are the n_1 l.i. columns of C, and the other columns are arbitrary vectors added such that **Q** is nonsingular.

Canonical Decompositions

Then equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\,\mathbf{x}$ partitions original state-space equations into:

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{\mathbf{c}} \\ \dot{\bar{\mathbf{x}}}_{\bar{\mathbf{c}}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{\mathbf{c}} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{\mathbf{c}}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{\mathbf{c}} \\ \bar{\mathbf{x}}_{\bar{\mathbf{c}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{\mathbf{c}} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t)$$

$$\bar{\mathbf{y}} = \begin{bmatrix} \bar{\mathbf{C}}_{\mathbf{c}} & \bar{\mathbf{C}}_{\bar{\mathbf{c}}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{\mathbf{c}} \\ \bar{\mathbf{x}}_{\bar{\mathbf{c}}} \end{bmatrix} + \mathbf{D}\,\mathbf{u}(t)$$

where $\bar{\mathbf{A}}_{\mathbf{c}}$ is n_1 x n_1 and $\bar{\mathbf{A}}_{\bar{\mathbf{c}}}$ is $(n\text{-}n_1)$ x $(n\text{-}n_1)$, and the n_1 dimensional subequation is controllable and has the same transfer matrix as the original state-space equation.

$$\dot{\bar{\mathbf{x}}}_{\mathbf{c}} = \bar{\mathbf{A}}_{\mathbf{c}}\bar{\mathbf{x}}_{\mathbf{c}} + \bar{\mathbf{B}}_{\mathbf{c}}\mathbf{u}(t)$$

$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_{\mathbf{c}}\bar{\mathbf{x}}_{\mathbf{c}} + \mathbf{D}\,\mathbf{u}(t)$$

Canonical Decompositions

Given the Observability matrix of an n dimensional system that is not observable:

$$\mathrm{O} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & \mathbf{A}^{2}{}'\mathbf{C}' & \mathbf{A}^{3}{}'\mathbf{C}' & \ldots & \mathbf{A}^{n-1}{}'\mathbf{C}' \end{bmatrix}'$$

where $\rho(\mathrm{O}) = n_2 < n$

Define equivalence transformation

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1{}' & \mathbf{p}_2{}' & \ldots & \mathbf{p}_{n_2}{}' & \ldots & \mathbf{p}_n{}' \end{bmatrix}' \qquad \mathbf{Q} = \mathbf{P}^{-1}$$

where the first n_2 rows of **P** are the n_2 l.i. rows of O, and the other rows are arbitrary vectors added such that **P** is nonsingular.

Canonical Decompositions

Then equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ partitions original state-space equations into:

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u}(t)$$

$$\bar{\mathbf{y}} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}(t)$$

where $\bar{\mathbf{A}}_o$ is n_2 x n_2 and $\bar{\mathbf{A}}_{\bar{o}}$ is $(n-n_2)$ x $(n-n_2)$, and the n_2 dimensional subequation is observable and has the same transfer matrix as the original state-space equation:

$$\dot{\bar{\mathbf{x}}}_o = \bar{\mathbf{A}}_o \bar{\mathbf{x}}_o + \bar{\mathbf{B}}_o \mathbf{u}(t)$$

$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_o \bar{\mathbf{x}}_o + \mathbf{D}\mathbf{u}(t)$$

Example Decomposition

Find the transformation to partition system below into observable and unobservable:

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t) \qquad y = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

Show result:

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} u(t) \qquad y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Kalman Decomposition Theorem

An equivalence transformation exists to transform any state-space equation into the following canonical form:

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t)$$

$$\bar{\mathbf{y}} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}(t)$$

where subscript *co* indicates the controllable and observable, and the bar over the subscript indicates *not*.

Kalman Decomposition Example

Perform a Kalman decomposition on:

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 \\ 0 & 1 & 0 & 0 & 0 & -74 \\ 0 & 0 & 1 & 0 & 0 & -85 \\ 0 & 0 & 0 & 1 & 0 & -45 \\ 0 & 0 & 0 & 0 & 1 & -11 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y} = \begin{bmatrix} 2 & -3 & 6 & -16 & 38 & -60 \end{bmatrix} \mathbf{x}$$

Kalman Decomposition Example

Previous example should yield :

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -12 & -31 & -27 & -9 & 0 & 0 \\ 0.0856 & 0.2884 & 0.1393 & 0.0199 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -3 \\ 6 \\ -16 \\ 38 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$

Kalman Decomposition Theorem

The controllable and observable subsystem is equivalent to the zero-state system given as:

$$\dot{\bar{\mathbf{x}}}_{co} = \bar{\mathbf{A}}_{co}\,\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\,\mathbf{u}(t)$$
$$\bar{\mathbf{y}} = \bar{\mathbf{C}}_{co}\,\bar{\mathbf{x}}_{co} + \mathbf{D}\,\mathbf{u}(t)$$

and has transfer matrix:

$$\hat{\mathbf{G}}(s) = \bar{\mathbf{C}}_{co}\left(s\mathbf{I} - \bar{\mathbf{A}}_{co}\right)^{-1}\hat{\mathbf{B}}_{co} + \mathbf{D}$$

Controllability (Discrete)

The state equation:

$$\mathbf{x}[k+1] = \mathbf{A}\,\mathbf{x}[k] + \mathbf{B}\,\mathbf{u}[k]$$

is controllable if for any pair of states $\mathbf{x}[k_1]$ and $\mathbf{x}[k_0]$, ∃ an input $\mathbf{u}[k]$ that drives state $\mathbf{x}[k_0]$ to $\mathbf{x}[k_1]$ in a finite number of samples.

If the system is controllable, then an input to transfer state $\mathbf{x}[k_0]$ to $\mathbf{x}[k_1]$ over the input samples in the interval $[k_0, k_1\text{-}1]$ is given by:

$$\mathbf{u}[k] = -\mathbf{B}'(\mathbf{A}')^{[k_1-k-1]}\mathbf{W}_{\mathbf{dc}}^{-1}[k_1-k_0-1]\left(\mathbf{A}^{[k_1-k_0]}\mathbf{x}[k_0]-\mathbf{x}[k_1]\right)$$

where

$$\mathbf{W}_{\mathbf{dc}}[n-1] = \sum_{m=0}^{n-1}\mathbf{A}^m\,\mathbf{B}\,\mathbf{B}'(\mathbf{A}')^m$$

Conditions for Controllability

For an n state and p input system: $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$

This system is controllable if any one of the equivalent conditions are met:

1. The n x n matrix $\mathbf{W}_{\mathbf{dc}}[n\text{-}1]$ is nonsingular

$$\mathbf{W}_{\mathbf{dc}}[n-1] = \sum_{m=0}^{n-1} \mathbf{A}^{m}\,\mathbf{B}\,\mathbf{B}'(\mathbf{A}')^{m}$$

2. The n x np controllability matrix C_d has full row rank (n):

$$C_d = \begin{bmatrix} \mathbf{B} & \mathbf{A}\,\mathbf{B} & \mathbf{A}^{2}\mathbf{B} & \mathbf{A}^{3}\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Conditions for Controllability

3. The n x $(n+p)$ matrix $[(\mathbf{A}-\lambda\mathbf{I}) \;\; \mathbf{B}]$ has full row rank for every eigenvalue λ of **A**.

4. All eigenvalues of **A** have magnitudes less than 1, and the unique solution $\mathbf{W}_{dc}$ is positive definite.

$$\mathbf{W}_{dc} - \mathbf{A}\,\mathbf{W}_{dc}\,\mathbf{A}' = \mathbf{B}\,\mathbf{B}'$$

where $\mathbf{W}_{dc}$ is the discrete controllability Gramian:

$$\mathbf{W}_{\mathbf{dc}} = \sum_{m=0}^{\infty} \mathbf{A}^{m}\,\mathbf{B}\,\mathbf{B}'(\mathbf{A}')^{m}$$

Observability (Discrete)

The discrete state-space equation:

$$\mathbf{x}[k+1] = \mathbf{A}\,\mathbf{x}[k] + \mathbf{B}\,\mathbf{u}[k] \qquad \mathbf{y}[k] = \mathbf{C}\,\mathbf{x}[k] + \mathbf{D}\,\mathbf{u}[k]$$

is observable if for any unknown initial state $\mathbf{x}[k_0]$, there exists a finite integer $k_1 - k_0$ э knowledge of input $\mathbf{u}[k]$ and output input $\mathbf{y}$[k] over $[k_0, k_1]$ is all that is required to uniquely determine $\mathbf{x}[k_0]$.

If the system is observable, then an estimator/observer to compute state $\mathbf{x}[k_0]$ from the input and output over the time interval $[k_0, k_1]$ is given by:

$$\mathbf{x}[k_0] = \left(\mathbf{W_{do}}[k_1 - k_0]\right)^{-1} \sum_{m=0}^{k_1 - k_0} (\mathbf{A'})^{\mathbf{m}} \mathbf{C'} \bar{\mathbf{y}}[k_0 + m, k_0]$$

$$\mathbf{W_{do}}[n-1] = \sum_{m=0}^{n-1} (\mathbf{A'})^{m} \mathbf{C'}\mathbf{C}\,\mathbf{A}^{\mathbf{m}}$$

$$\bar{\mathbf{y}}[k, k_0] = \mathbf{y}[k] - \mathbf{C} \sum_{m=k_0}^{k-1} \mathbf{A}^{k-1-m} \mathbf{B}\,\mathbf{u}[m] - \mathbf{D}\mathbf{u}[k]$$

Conditions for Observability

For an *n* state and *q* output system:

$$\mathbf{x}[k+1] = \mathbf{A}\,\mathbf{x}[k] + \mathbf{B}\,\mathbf{u}[k] \qquad \mathbf{y}[k] = \mathbf{C}\,\mathbf{x}[k] + \mathbf{D}\,\mathbf{u}[k]$$

1. This system is observable iff the *n*x*n* matrix $\mathbf{W_{do}}$ is nonsingular

$$\mathbf{W_{do}}[k_1 - 1] = \sum_{m=0}^{k_1-1} \mathbf{A}^m \mathbf{C}\,\mathbf{C}'(\mathbf{A}')^m$$

2. The *nq* x *n* observability matrix O_d has full column rank (*n*):

$$O_d = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\,\mathbf{A} \\ \mathbf{C}\,\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\,\mathbf{A}^{n-1} \end{bmatrix}$$

Conditions for Observability

3. The $(n+q)$ x n matrix $[(\mathbf{A}-\lambda\mathbf{I})' \ \ \mathbf{C}']'$ has full column rank for every eigenvalue λ of **A**.

4. All eigenvalues of **A** have magnitudes less than 1, and the unique solution $\mathbf{W}_{do}$ is positive definite

$$\mathbf{W}_{do} - \mathbf{A}'\mathbf{W}_{do}\mathbf{A} = \mathbf{C}'\mathbf{C}$$

where $\mathbf{W}_{do}$ is the observability Gramian:

$$\mathbf{W}_{do} = \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{C}'\mathbf{C}\mathbf{A}^m$$

Controllability After Sampling

Given a controllable continuous-time system, a sufficient condition for controllability of its discretized system using sampling period T is that:

$\mathbf{Im}[\lambda_i - \lambda_j] \neq 2\pi m / T$ for any *ij* pair of eigenvalues from continuous time system whenever $\mathbf{Re}[\lambda_i - \lambda_j] = 0$.

For the single-input system the above condition is necessary as well.

Homework U9.1

Write a Matlab code to compute the initial state x[0] and all subsequent states corresponding to the input-output pair sequence the interval $n \in [0\ 6]$ and the discrete-time system:

n	u[n]	y[n]
0	1	2
1	1	10.75
2	1	0.3675
3	0	1.1969
4	0	3.2546
5	0	-0.5657
6	0	-1.7105

$$\dot{\mathbf{x}} = \begin{bmatrix} 0.1 & -0.8 & 0 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0 \\ 0 & 0.75 & -0.2 & -0.75 & 0 \\ 0.8 & 0.3 & 0.75 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 0.5 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \end{bmatrix} \mathbf{x}$$

Indicate the state values for n=0 to 6 and hand in printout of commented code used to solve the problem.

Homework U9.2

Perform a Kalman decomposition on the system below

$$\dot{\mathbf{x}} = \begin{bmatrix} 0.1 & -0.8 & 0 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0 \\ 0 & 0.75 & -0.2 & -0.75 & 0 \\ 0.8 & 0.3 & 0.75 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y} = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

Indicate the A,B, and C matrices for the decomposed systems and write out the subsystem that is both controllable and observable. Hand in printout of commented code used to solve the problem.