EE611
Deterministic Systems

Controllability and Observability Discrete Systems

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Canonical Decompositions

Given the Controllability matrix of an $n$ dimensional system that is not controllable:

$$C = \begin{bmatrix} B & A B & A^2 B & A^3 B & \ldots & A^{n-1} B \end{bmatrix} \text{ where } \rho(C) = n_1 < n$$

Define equivalence transformation

$$Q = \begin{bmatrix} q_1 & q_2 & \ldots & q_{n_1} & \ldots & q_n \end{bmatrix} \quad P = Q^{-1}$$

where the first $n_1$ columns of $Q$ are the $n_1$ l.i. columns of $C$, and the other columns are arbitrary vectors added such that $Q$ is nonsingular.
Canonical Decompositions

Then equivalence transformation $\bar{x} = P x$ partitions original state-space equations into:

\[
\begin{bmatrix}
\dot{\bar{x}}_c \\
\dot{\bar{x}}_{\bar{c}}
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_c & \bar{A}_{12} \\
0 & \bar{A}_{\bar{c}}
\end{bmatrix}
\begin{bmatrix}
\bar{x}_c \\
\bar{x}_{\bar{c}}
\end{bmatrix} +
\begin{bmatrix}
\bar{B}_c \\
0
\end{bmatrix} u(t)
\]

\[
\bar{y} =
\begin{bmatrix}
\bar{C}_c & \bar{C}_{\bar{c}}
\end{bmatrix}
\begin{bmatrix}
\bar{x}_c \\
\bar{x}_{\bar{c}}
\end{bmatrix} + D u(t)
\]

where $\bar{A}_c$ is $n_1 \times n_1$ and $\bar{A}_{\bar{c}}$ is $(n-n_1) \times (n-n_1)$, and the $n_1$ dimensional subequation is controllable and has the same transfer matrix as the original state-space equation.

\[
\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u(t)
\]

\[
\bar{y} = \bar{C}_c \bar{x}_c + D u(t)
\]
Canonical Decompositions

Given the Observability matrix of an $n$ dimensional system that is not observable:

$$O = [C' \ A'\ C' \ A^2'\ C' \ A^3'\ C' \ \ldots \ A^{n-1}'\ C']'$$

where $\rho(O) = n_2 < n$

Define equivalence transformation

$$P = [p_1' \ p_2' \ \ldots \ p_{n_2}' \ \ldots \ p_n']' \quad Q = P^{-1}$$

where the first $n_2$ rows of $P$ are the $n_2$ l.i. rows of $O$, and the other rows are arbitrary vectors added such that $P$ is nonsingular.
Canonical Decompositions

Then equivalence transformation $\bar{x} = P x$ partitions original state-space equations into:

$$
\begin{bmatrix}
\dot{\bar{x}}_o \\
\dot{\bar{x}}_\bar{o}
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_o & 0 \\
\bar{A}_{21} & \bar{A}_{\bar{o}}
\end{bmatrix}
\begin{bmatrix}
\bar{x}_o \\
\bar{x}_\bar{o}
\end{bmatrix} +
\begin{bmatrix}
\bar{B}_o \\
\bar{B}_{\bar{o}}
\end{bmatrix} u(t)
$$

$$
\bar{y} =
\begin{bmatrix}
\bar{C}_o & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x}_o \\
\bar{x}_\bar{o}
\end{bmatrix} + D u(t)
$$

where $\bar{A}_o$ is $n_2 \times n_2$ and $\bar{A}_\bar{o}$ is $(n-n_2) \times (n-n_2)$, and the $n_2$ dimensional subequation is observable and has the same transfer matrix as the original state-space equation:

$$
\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u(t)
$$

$$
\bar{y} = \bar{C}_o \bar{x}_o + D u(t)
$$
Example Decomposition

Find the transformation to partition system below into observable and unobservable:

\[
\dot{x} = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -3 \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
1 \\
1 \\
\end{bmatrix} u(t)
\]

\[
y = \begin{bmatrix}
0 & 1 & 1 & 0 \\
\end{bmatrix} x
\]

Show result:

\[
\dot{\tilde{x}} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-2 & -3 & 0 & 0 \\
2 & 1 & -1 & 0 \\
0 & 0 & 0 & -3 \\
\end{bmatrix} \tilde{x} + \begin{bmatrix}
1 \\
-2 \\
1 \\
1 \\
\end{bmatrix} u(t)
\]

\[
y = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\end{bmatrix} \tilde{x}
\]
Kalman Decomposition Theorem

An equivalence transformation exists to transform any state-space equation into the following canonical form:

\[
\begin{bmatrix}
\dot{\bar{x}}_{\text{co}} \\
\dot{\bar{x}}_{\text{c}\bar{o}} \\
\dot{\bar{x}}_{\text{co}} \\
\dot{\bar{x}}_{\text{c}\bar{o}}
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_{\text{co}} & 0 & \bar{A}_{13} & 0 \\
\bar{A}_{21} & \bar{A}_{\text{co}} & \bar{A}_{23} & \bar{A}_{24} \\
0 & 0 & \bar{A}_{\text{c}\bar{o}} & 0 \\
0 & 0 & \bar{A}_{43} & \bar{A}_{\text{c}\bar{o}}
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{\text{co}} \\
\bar{x}_{\text{c}\bar{o}} \\
\bar{x}_{\text{co}} \\
\bar{x}_{\text{c}\bar{o}}
\end{bmatrix} +
\begin{bmatrix}
\bar{B}_{\text{co}} \\
\bar{B}_{\text{c}\bar{o}} \\
0 \\
0
\end{bmatrix} u(t)
\]

\[
\bar{y} = \begin{bmatrix}
\bar{C}_{\text{co}} & 0 & \bar{C}_{\text{c}\bar{o}} & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{\text{co}} \\
\bar{x}_{\text{c}\bar{o}} \\
\bar{x}_{\text{co}} \\
\bar{x}_{\text{c}\bar{o}}
\end{bmatrix} + D u(t)
\]

where subscript \(\text{co}\) indicates the controllable and observable, and the bar over the subscript indicates not.
Kalman Decomposition Example

Perform a Kalman decomposition on:

\[
\dot{x} = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -24 \\
0 & 1 & 0 & 0 & 0 & -74 & 0 \\
0 & 0 & 1 & 0 & 0 & -85 & 0 \\
0 & 0 & 0 & 1 & 0 & -45 & 0 \\
0 & 0 & 0 & 0 & 1 & -11 & 0 \\
\end{bmatrix}
x + \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} u(t)
\]

\[
y = \begin{bmatrix}
2 & -3 & 6 & -16 & 38 & -60
\end{bmatrix} x
\]
Kalman Decomposition Example

Previous example should yield:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-12 & -31 & -27 & -9 & 0 & 0 & 0 \\
0.0856 & 0.2884 & 0.1393 & 0.0199 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 \\
\end{bmatrix}
\begin{bmatrix}
x \\ u(t)
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 2
\end{bmatrix} x
\]
Kalman Decomposition Theorem

The controllable and observable subsystem is equivalent to the zero-state system given as:

\[
\dot{x}_{co} = \bar{A}_{co} x_{co} + \bar{B}_{co} u(t)
\]

\[
y = \bar{C}_{co} x_{co} + D u(t)
\]

and has transfer matrix:

\[
\hat{G}(s) = \bar{C}_{co} \left( s I - \bar{A}_{co} \right)^{-1} \hat{B}_{co} + D
\]
Controllability (Discrete)

The state equation:

\[ x[k+1] = A x[k] + B u[k] \]

is controllable if for any pair of states \( x[k_1] \) and \( x[k_0] \), \( \exists \) an input \( u[k] \) that drives state \( x[k_0] \) to \( x[k_1] \) in a finite number of samples.

If the system is controllable, then an input to transfer state \( x[k_0] \) to \( x[k_1] \) over the input samples in the interval \([k_0, k_1-1]\) is given by:

\[
u[k] = -B'(A')^{k_1-k_0-1} W_{dc}^{-1}[k_1-k_0-1] \left( A^{k_1-k_0} x[k_0] - x[k_1] \right)\]

where

\[
W_{dc}[n-1] = \sum_{m=0}^{n-1} A^m B B'(A')^m
\]
Conditions for Controllability

For an $n$ state and $p$ input system: $\dot{x} = Ax + Bu$

This system is controllable if any one of the equivalent conditions are met:

1. The $n \times n$ matrix $W_{dc}[n-1]$ is nonsingular

$$W_{dc}[n-1] = \sum_{m=0}^{n-1} A^m B B'(A')^m$$

2. The $n \times np$ controllability matrix $C_d$ has full row rank ($n$):

$$C_d = [B \ A \ B \ A^2 B \ A^3 B \ ... \ A^{n-1} B]$$
3. The $n \times (n+p)$ matrix $[(A-\lambda I) B]$ has full row rank for every eigenvalue $\lambda$ of $A$.

4. All eigenvalues of $A$ have magnitudes less than 1, and the unique solution $W_{dc}$ is positive definite.

$$W_{dc} - A W_{dc} A' = B B'$$

where $W_{dc}$ is the discrete controllability Gramian:

$$W_{dc} = \sum_{m=0}^{\infty} A^m B B' (A')^m$$
Observability (Discrete)

The discrete state-space equation:
\[
x[k+1] = Ax[k] + Bu[k] \quad y[k] = Cx[k] + Du[k]
\]
is observable if for any unknown initial state \(x[k_0]\), there exists a finite integer \(k_1 - k_0 \in \mathbb{N}\) knowledge of input \(u[k]\) and output input \(y[k]\) over \([k_0, k_1]\) is all that is required to uniquely determine \(x[k_0]\).

If the system is observable, then an estimator/observer to compute state \(x[k_0]\) from the input and output over the time interval \([k_0, k_1]\) is given by:
\[
x[k_0] = \left(W_{do}[k_1 - k_0]\right)^{-1} \sum_{m=0}^{k_1 - k_0} (A')[m]C'y[k_0 + m, k_0]
\]
\[
W_{do}[n-1] = \sum_{m=0}^{n-1} (A')[m]C'C A^m
\]
\[
\bar{y}[k, k_0] = y[k] - C \sum_{m=k_0}^{k-1} A^{k-1-m} Bu[m] - Du[k]
\]
Conditions for Observability

For an $n$ state and $q$ output system:

\[
x[k+1] = Ax[k] + Bu[k] \quad y[k] = Cx[k] + Du[k]
\]

1. This system is observable iff the $nxn$ matrix $W_{do}$ is nonsingular

\[
W_{do}[k_1-1] = \sum_{m=0}^{k_1-1} A^m C C' (A')^m
\]

2. The $nq \times n$ observability matrix $O_d$ has full column rank ($n$):

\[
O_d = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]
Conditions for Observability

3. The \((n+q) \times n\) matrix \([(A-\lambda I)' \ C' \ A^{-\lambda I}]'\)' has full column rank for every eigenvalue \(\lambda\) of \(A\).

4. All eigenvalues of \(A\) have magnitudes less than 1, and the unique solution \(W_{do}\) is positive definite

\[ W_{do} - A' W_{do} A = C' C \]

where \(W_{do}\) is the observability Gramian:

\[ W_{do} = \sum_{m=0}^{\infty} (A')^m C'C A^m \]
Controllability After Sampling

Given a controllable continuous-time system, a sufficient condition for controllability of its discretized system using sampling period $T$ is that:

$$\text{Im}[\lambda_i - \lambda_j] \neq 2\pi m / T$$ for any $ij$ pair of eigenvalues from continuous time system whenever $\text{Re}[\lambda_i - \lambda_j] = 0$.

For the single-input system the above condition is necessary as well.
Homework U9.1

Write a Matlab code to compute the initial state $x[0]$ and all subsequent states corresponding to the input-output pair sequence the interval $n \in [0 \ 6]$ and the discrete-time system:

$$
\dot{x} = \begin{bmatrix}
0.1 & -0.8 & 0 & 0 & 0 \\
0.8 & 0.1 & 0 & 0 & 0 \\
0 & 0.75 & -0.2 & -0.75 & 0 \\
0.8 & 0.3 & 0.75 & -0.2 & 0 \\
0 & 0 & 0 & 0 & 0.5 \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
1 \\
1 \\
2 \\
0.5 \\
\end{bmatrix} u(t)
$$

$$
y = \begin{bmatrix}
1 & 0 & 1 & 1 & 2
\end{bmatrix} x
$$

Indicate the state values for $n=0$ to $6$ and hand in printout of commented code used to solve the problem.
Homework U9.2

Perform a Kalman decomposition on the system below

\[
\begin{bmatrix}
0.1 & -0.8 & 0 & 0 & 0 \\
0.8 & 0.1 & 0 & 0 & 0 \\
0 & 0.75 & -0.2 & -0.75 & 0 \\
0.8 & 0.3 & 0.75 & -0.2 & 0 \\
0 & 0 & 0 & 0 & 0.5 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
1 \\
0 \\
2 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t) \\
u_4(t) \\
u_5(t) \\
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
0 & -1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
\]

Indicate the A, B, and C matrices for the decomposed systems and write out the subsystem that is both controllable and observable. Hand in printout of commented code used to solve the problem.