EE611
Deterministic Systems

Multiple-Input Multiple-Output (MIMO) Feedback

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MIMO State Feedback

Consider designing state feedback for a MIMO system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x} + \mathbf{E}\mathbf{u}(t)$$

Feedback design results in matrix **K**

$$\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x}$$

and the feedback compensated system becomes:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{r}(t)$$
$$\mathbf{y}(t) = (\mathbf{C} - \mathbf{E}\mathbf{K})\mathbf{x} + \mathbf{E}\mathbf{r}(t)$$

Design by Converting to Single Input System

Matrix **A** is cyclic iff its characteristic polynomial is equal to its minimal polynomial (*i.e.* one and only one Jordan block for each eigenvalue)

Theorem: If {**A**, **B**} is controllable and **A** is cyclic, then for *almost* any px1 real vector **v** the single input pair {**A**, **Bv** } is controllable.

"*Almost*" means rows of **Bv** corresponding to the last row of each Jordan block are not zero.

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ \alpha \\ b \\ \beta \end{bmatrix} \qquad \alpha \neq 0 \qquad \beta \neq 0$$

Creating Cyclic with Feedback

Theorem: If {**A**, **B**} is controllable, then for almost any *p*x*n* real constant matrix **K**, all the eigenvalues of **A**–**BK** are *distinct* and consequently (**A-BK**) is cyclic.

Theorem: If {**A**, **B**} is controllable, then by state feedback of the form **u** = **r** - **Kx** where **K** is a *p*x*n* real constant matrix, the eigenvalues of **A-BK** can be arbitrarily assigned.

Feedback design process follows the proof ... If **A** not cyclic, make it cyclic with an arbitrary feedback matrix, then design another feedback matrix/vector to place eigenvalues at desired positions.

Example

Convert Problem to single input system to design a feedback matrix **K** to place eigenvalues at -2, -1.5±j1.5, -1±j2

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

For reduction vector **v** = $[1\ 1]^T$, show

$$\mathbf{K} = \begin{bmatrix} 19.4 & -19.5 & 45 & -12.5 & -15.8 \\ 19.4 & -19.5 & 45 & -12.5 & -15.8 \end{bmatrix}$$

Design Using Controllable Canonical Form

Given controllable {**A**, **B**} with

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_p \end{bmatrix}$$

Create initial controllability matrix:

$$\mathrm{M}' = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \ldots & \mathbf{A}^{n-\bar{p}}\mathbf{B} \end{bmatrix}$$

where $\bar{p}$ is the rank of **B**. Obtain a non-singular square matrix M by finding the l.i. columns from left to right of M' and rearrange to group together columns of **B**:

$$\mathrm{M} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{A}\mathbf{b}_1 & \ldots & \mathbf{A}^{\mu_1-1}\mathbf{b}_1 & \vdots & \mathbf{b}_2 & \mathbf{A}\mathbf{b}_2 & \ldots & \mathbf{A}^{\mu_2-1}\mathbf{b}_2 & \vdots & \ldots & \vdots & \mathbf{b}_{\bar{p}} & \ldots & \mathbf{A}^{\mu_{\bar{p}}-1}\mathbf{b}_{\bar{p}} \end{bmatrix}$$

where μ_i are the controllability indices associated with each l.i. column of **B**.

Take inverse of M and partition rows according to blocks associated with each $\mathbf{b}_i$:

$$\mathrm{M}^{-1} = \begin{bmatrix} \mathbf{e}_{11}^{\mathrm{T}} & \mathbf{e}_{21}^{\mathrm{T}} & \ldots & \mathbf{e}_{\mu_1 1}^{\mathrm{T}} & \vdots & \mathbf{e}_{12}^{\mathrm{T}} & \mathbf{e}_{22}^{\mathrm{T}} & \ldots & \mathbf{e}_{\mu_2 2}^{\mathrm{T}} & \vdots & \ldots & \vdots & \mathbf{e}_{1\bar{\mathrm{p}}}^{\mathrm{T}} & \ldots & \mathbf{e}_{\mu_{\bar{p}}\,\bar{\mathrm{p}}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

Design Using Controllable Canonical Form

Use last rows of each partition in M-1 to create P matrix for transformation to controllable canonical form:

$$\mathbf{P}=\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{e}_{\mu_{\bar{p}}\,\bar{\mathbf{p}}} \\ \mathbf{e}_{\mu_{\bar{p}}\,\bar{\mathbf{p}}}\mathbf{A} \\ \dots \\ \mathbf{e}_{\mu_{\bar{p}}\,\bar{\mathbf{p}}}\mathbf{A}^{\mu_{\bar{p}}-1} \\ \dots \\ \vdots \\ \dots \\ \mathbf{e}_{\mu_1 1} \\ \mathbf{e}_{\mu_1 1}\mathbf{A} \\ \vdots \\ \mathbf{e}_{\mu_1 1}\mathbf{A}^{\mu_1-1} \end{bmatrix}$$

MIMO Controllable Canonical Form

$$\mathbf{A}=\begin{bmatrix}
-\alpha_{\bar p 1} & -\alpha_{\bar p 2} & \cdots & -\alpha_{\bar p \mu_{\bar p}-2} & -\alpha_{\bar p \mu_{\bar p}-1} & \vdots & \cdots & \vdots & x & x & \cdots & x & x \\
1 & 0 & \cdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x & x & \cdots & x & x & \vdots & \cdots & \vdots & -\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1\mu_1-2} & -\alpha_{1\mu_1-1} \\
0 & 0 & \cdots & 0 & 0 & \vdots & \cdots & \vdots & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\qquad
\mathbf{B}=\begin{bmatrix}
1 & \cdots & x \\
0 & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 1 \\
0 & \cdots & 0 \\
\vdots & \vdots & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}$$

where x implies any number

Example

Convert the MIMO system below to a controllable canonical form:

$$\dot{\mathbf{x}} = \begin{bmatrix} -.4 & .5 & -22.2 & -32.1 & -11.2 & 12.3 \\ 0 & -.4 & 18.8 & 8.2 & 0 & 10.6 \\ 0 & 0 & -21 & -25 & 8 & 6 \\ 0 & 0 & 20 & 19 & -8 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & -8 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4.8792 & 0 \\ -4.8792 & 1 \\ -.6099 & 0 \\ 1.2198 & -1 \\ -1.2198 & 0 \\ 2.4396 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0.1007 & 0.0349 & -0.7804 & 0.0349 & 6.0808 & -1.4028 \\ -0.0184 & 0.0203 & 8.7429 & 10.0203 & 0.9200 & -2.2871 \end{bmatrix} \mathbf{x}$$

Example

Design a feedback matrix K so that resulting eigenvalues are at -1, -2±j, and -1±j2

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & -.5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} -1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \mathbf{x}$$

Lyapunov Method MIMO

Find equivalence transformation such that desired **K** vector satisfies:

$$\mathbf{T}^{-1}(\mathbf{A}-\mathbf{B}\mathbf{K})\mathbf{T}=\mathbf{F}$$

where **F** is a matrix with desired eigenvalues distinct from **A** (i.e. the eigenvalues of the feedback system).

Terms can be rearranged to show:

$$\mathbf{A}\mathbf{T}-\mathbf{T}\mathbf{F}=\mathbf{B}\mathbf{K}\mathbf{T}$$

$$\mathbf{A}\mathbf{T}-\mathbf{T}\mathbf{F}=\mathbf{B}\bar{\mathbf{K}}$$

Therefore, $\bar{\mathbf{K}}$ can be chosen almost arbitrarily, and similarity transform matrix **T** solved for to obtain

$$\mathbf{K}=\bar{\mathbf{K}}\mathbf{T}^{-1}$$

Lyapunov Method MIMO

If **A** and **F** have no eigenvalues in common, then a solution for **T** exists in $\mathbf{AT}-\mathbf{TF}=\mathbf{B}\bar{\mathbf{K}}$ and is nonsingular iff (**A**, **B**) is controllable and ($\mathbf{F},\bar{\mathbf{K}}$) observable.

Procedure:

1. Select *n* x *n* matrix **F** with desired eigenvalues.
2. Select arbitrary *p* x *n* vector $\bar{\mathbf{l}}$ such that ($\mathbf{F},\bar{\mathbf{l}}$) controllable.
3. Solve for **T** in the Lyapunov equation $\mathbf{AT}-\mathbf{TF}=\mathbf{B}\bar{\mathbf{K}}$
4. Compute feedback gain matrix $\mathbf{K}=\bar{\mathbf{K}}\mathbf{T}^{-1}$

State Observer/Estimators

The extension from SISO to MIMO systems for state observers is trivial for the Lyapunov method. Extend single row vector **c** to **C**, and observer gain vector **l** to **L.**

For the reduced state estimator, the observer can be reduced to a dimension of $n - \rho(\mathbf{C})$.