Fourier Transform Properties

The following properties describe the relationship between the time and frequency domains.

Duality: If \( f(t) \Leftrightarrow F(\omega) \), then \( F(t) \Leftrightarrow 2\pi f(-\omega) \)

Example: Apply duality to \( \exp(j\omega_0 t) \Leftrightarrow 2\pi \delta(\omega - \omega_0) \)

Scaling Property: If \( f(t) \Leftrightarrow F(\omega) \), then for any real constant \( a \), \( f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \)

Example: What happens to the spectrum of a sound on a tape recording if it is played back at a faster speed?

Time-Shifting Property: If \( f(t) \Leftrightarrow F(\omega) \), then \( f(t - t_0) \Leftrightarrow F(\omega) \exp(-j\omega t_0) \)

Example: Prove

Frequency-Shifting Property: If \( f(t) \Leftrightarrow F(\omega) \), then \( f(t) \exp(j\omega_0 t) \Leftrightarrow F(\omega - \omega_0) \)

Example: Show what happens to the signal spectrum when modulated by a sinusoid.

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Convolution: If \( f_1(t) \Leftrightarrow F_1(\omega) \) and \( f_2(t) \Leftrightarrow F_2(\omega) \), then convolution in one domain is multiplication in the other domain: 
\[
 f_1(t) \ast f_2(t) \Leftrightarrow F_1(\omega)F_2(\omega) 
\]
and
\[
f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi}F_1(\omega)F_2(\omega)
\]
Example: Prove time-convolution part.

Time Differentiation: If \( f(t) \Leftrightarrow F(\omega) \), then
\[
 \frac{df(t)}{dt} \Leftrightarrow j\omega F(\omega) \quad \text{and} \quad \frac{d^n f(t)}{dt^n} \Leftrightarrow (j\omega)^n F(\omega)
\]
Example: Explain what happens to the DC component of a signal when the derivative is taken in the time domain. What happens to the high frequency components?

Time Integration: If \( f(t) \Leftrightarrow F(\omega) \), then
\[
 \int_{-\infty}^{t} f(\tau) d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + j\pi F(0) \delta(\omega)
\]
Example: Explain what happens to the spectrum of an integrated signal when a DC component is present? What happens to the high frequency components?
Signal Distortion:

Distortion occurs when the magnitude and/or phase spectrum of the signal is modified in some way. In many applications distortion is not a good thing, as in the case of signal measurement, audio and video reproduction, amplification, and communication. Distortion exists because every device used to measure, transmit, process, and/or display electrical signals has a non-flat transfer function magnitude and a non-linear phase.

Example: Consider a modulated square pulse \( s(t) \) used to transmit binary information down a bandlimited channel with transfer function \( H(j\omega) \).

\[
    s(t) = \text{rect}\left(\frac{t}{\tau}\right)\cos(\omega_0 t)
    \quad
    H(j\omega) = \frac{(2\pi 100)p}{p^2 + (2\pi 100)p + (2\pi 500)^2}
\]

Plot the output of this channel for \( \tau = 1/50 \), \( \tau = 1/100 \), and \( \tau = 1/300 \).
Try this for \( \omega_0 = 2\pi 500 \) rads/sec.

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The FT of the pulse is (use table): \[ \text{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \ \text{sinc}\left(\frac{\omega \tau}{2}\right) \]

For modulate pulse apply the frequency shift property (use Euler's identity and table):
\[ \text{rect}\left(\frac{t}{\tau}\right)\left(\frac{\exp(j\omega_0 t) + \exp(-j\omega_0 t)}{2}\right) \Leftrightarrow \frac{\tau}{2} \text{sinc}\left(\frac{(\omega - \omega_0) \tau}{2}\right) + \frac{\tau}{2} \text{sinc}\left(\frac{(\omega + \omega_0) \tau}{2}\right) \]

Plot spectrum for transfer function and the 3 versions of the pulse:

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To obtain the time domain signal that represents the output of the channel, take the product of the FT of the signal and the transfer function, (i.e. each frequency component of the transfer function scales and phase shifts the corresponding frequency components of the signal), and take the inverse FT of the product.

\[ y(t) = \int_{-\infty}^{\infty} \left[ \text{sinc}\left(\frac{(\omega - \omega_0)\tau}{2}\right) + \text{sinc}\left(\frac{(\omega + \omega_0)\tau}{2}\right) \right] \frac{2\pi 100(j\omega)}{-\omega^2 + 2\pi 100(j\omega) - (2\pi 500)^2} \exp(j\omega t) d\omega \]

After examining the integral, it can be determine that its evaluation is a significant amount of tedious work. In this case it would be better to do numerically (i.e. let Matlab take the Fourier Transforms (see help on fft and ifft).

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original signal, $\tau = 1/50$ seconds

original signal, $\tau = 1/100$ seconds

original signal, $\tau = 1/300$ seconds

channel output, $\tau = 1/50$ seconds

channel output, $\tau = 1/100$ seconds

channel output, $\tau = 1/300$ seconds

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Matlab Code that Generated Distortion Plots:

tint = .03;                           % time interval over which pulse is defined
fs = 10e3;                           % Set sampling Rate to about 10 times highest pulse freq.
t = [-tint/2:(1/fs):(tint)];         % Define t-axis with a sampling rate fs
len = length(t)                      % See how big time axis vector is
tau1 = 1/50;                        % Pulse duration case 1
tau2 = 1/100;                       % Pulse duration case 2
tau3 = 1/300;                       % Pulse duration case 3
w0 = 500*2*pi;                      % Modulating frequency and center
% frequency of Transfer function
a = 2*pi*100;                       % Transfer function gain parameter (a/b)
b = 2*pi*100;                       % Transfer function bandwidth parameter

figure(1);                          % Case 1 pulse
s1 = tau1*rectw(t,tau1).*cos(w0*t);  % create signal
plot(t,s1);
title('original signal, tau = 1/50')
xlabel('seconds')
grid

fs1 = fft(s1,2*len);                 % Take the FT to create twice the frequency samples
f = [-fs/2:(fs/(2*len)):(fs/2)-fs/(2*len)]; % Create Frequency Axis to
% evaluate Transfer function
p = j*2*pi*f;                       % Create jw values
h = a*p ./ ( p.^2+b*p+w0^2);         % Evaluate transfer function
pd = fs1.*fftshift(h);              % Must shift Frequency axis, multiply by
% FT of pulse, and divide by interval
% over which the pulse was defined

figure(2)
y1 = real(ifft(pd));                 % Take inverse FT (then take real part).
plot(t,y1(1:len));
title('channel output, tau = 1/50')
xlabel('seconds')
grid

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figure(1);                        % Case 2 pulse
s2 = tau2*rectw(t,tau2).*cos(w0*t);
plot(t,s2);
title('original signal, tau = 1/100')
xlabel('seconds')
grid
fs2 = fft(s2,2*len);            % Take the FT to create twice the frequency samples
p = j*2*pi*f;                  % Create jw values
h = a*p ./ ( p.^2+b*p+w0^2);   % Evaluate transfer function
pd = fs2.*fftshift(h);         % Must shift Frequency axis, multiply by
% FT of pulse, and divide by interval
% over which the pulse was defined
figure(2) y2 = real(ifft(pd));           % Take inverse FT (then take real part).
plot(t,y2(1:len));
title('channel output, tau = 1/100')
xlabel('seconds')
grid

figure(1);                        % Case 3 pulse
s3 = tau3*rectw(t,tau3).*cos(w0*t);
plot(t,s3);
title('original signal, tau = 1/300')
xlabel('seconds')
grid
fs3 = fft(s3,2*len);            % Take the FT to create twice the frequency samples
p = j*2*pi*f;                  % Create jw values
h = a*p ./ ( p.^2+b*p+w0^2);   % Evaluate transfer function
pd = fs3.*fftshift(h);         % Must shift Frequency axis, multiply by
% FT of pulse, and divide by interval
% over which the pulse was defined
figure(2) y3 = real(ifft(pd));           % Take inverse FT (then take real part).
plot(t,y3(1:len));

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Energy/Power and Parseval's Theorem

For a periodic signal the average power in the time domain is equal to the sum of the Fourier Series coefficients squared:

\[ P_{av} = \frac{1}{T} \int_{t} |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |D_n|^2 \]

Example: How is this result expressed in terms of the other Fourier Series representations?

For an energy signal (nonperiodic) the energy in the time domain is equal to the area under the Fourier Transform magnitude squared (scaled by 2\(\pi\)):

\[ E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \]

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Example: How much energy is lost when a square pulse of duration .01 seconds is passed through and ideal low-pass filter with cutoff frequency of 100 Hz?