ANALYSIS OF “POOR MAN’S NAVIER–STOKES”
AND THERMAL ENERGY EQUATIONS FOR
HIGH-RAYLEIGH NUMBER TURBULENT CONVECTION

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Efficient LES Models Able to Account for Interactions of Physics on Small Scales Still in Their Infancy

Physically-Realistic Temporal Fluctuations Required for Such Models

Implies Need to Retain as Much of Governing Equations as Possible within Framework of Efficient Model

Possible Approach: use symbol (or, more accurately, a pseudo-differential operator) of governing equations

Present Work Describes Way to Do This, and Demonstrates Outcomes in Terms of Comparisons with Actual Physics
OUTLINE OF PMNS + THERMAL ENERGY EQ. DERIVATION

  - Invoke **Leray projection** to remove $\nabla p$ from momentum equations
  - Assume **Fourier representations** for dependent variables $u, v, T$ with basis functions exhibiting complex exponential-like differentiation properties
  - Construct **Galerkin ODEs**

\[
\begin{align*}
\dot{a}_k &= -\sum_{\ell,m=1}^{\infty} A_{\ell m k}^{(1)} a_{\ell}^2 + A_{\ell m k}^{(2)} a_m b_{\ell} - \frac{|k|^2}{Re} a_k \\
\dot{b}_k &= -\sum_{\ell,m=1}^{\infty} B_{\ell m k}^{(1)} b_{\ell}^2 + B_{\ell m k}^{(2)} a_m b_{\ell} - \frac{|k|^2}{Re} b_k - \frac{Gr}{Re} c_k \\
\dot{c}_k &= -\sum_{\ell,m=1}^{\infty} C_{\ell m k}^{(1)} a_m c_{\ell} + C_{\ell m k}^{(2)} b_m c_{\ell} - \frac{|k|^2}{Pe} c_k, \quad k = 1, \ldots, \infty
\end{align*}
\]
Decimate infinite system to single wavevector

\[ \dot{a}_k + A_{lmk}^{(1)} a_k^2 + A_{lmk}^{(2)} a_k b_k = -\frac{|k|^2}{Re} a_k \]

\[ \dot{b}_k + B_{lmk}^{(1)} b_k^2 + B_{lmk}^{(2)} a_k b_k = -\frac{|k|^2}{Re} b_k - \frac{Gr}{Re^2} c_k \]

\[ \dot{c}_k + C_{lmk}^{(1)} a_k c_k + C_{lmk}^{(2)} b_k c_k = -\frac{|k|^2}{Pe} c_k \]

Remark: right-hand sides are symbols, and left-hand sides similar to pseudo-differential operators

Use simple Euler time integrators (and suppress indices)

\[ a^{(n+1)} = a^{(n)} - \tau \left[ \frac{|k|^2}{Re} a^{(n)} + A^{(1)} (a^{(n)})^2 + A^{(2)} a^{(n)} b^{(n)} \right] \]

\[ b^{(n+1)} = b^{(n)} - \tau \left[ \frac{|k|^2}{Re} b^{(n)} + B^{(1)} (b^{(n)})^2 + B^{(2)} a^{(n)} b^{(n)} + \frac{Gr}{Re^2} c \right] \]

\[ c^{(n+1)} = c^{(n)} - \tau \left[ C^{(1)} a^{(n)} c^{(n)} + C^{(2)} b^{(n)} c^{(n)} \right] / \left( 1 + \frac{|k|^2}{Pe} \right) \]
**PMNS/Thermal Energy Eq. Derivation (Cont.)**


  \[ a^{(n+1)} = \beta_1 a^{(n)}(1-a^{(n)}) - \gamma_{12} a^{(n)} b^{(n)} \]

  \[ b^{(n+1)} = \beta_2 b^{(n)}(1-b^{(n)}) - \gamma_{21} a^{(n)} b^{(n)} + \alpha_T c^{(n)} \]

  \[ c^{(n+1)} = -\left( \gamma_{uT} a^{(n+1)} + \gamma_{vT} b^{(n+1)} \right) c^{(n)} / (1+\beta_T) + c_0 \]

- **Total of 9 bifurcation parameters:** \( \beta_1, \beta_2, \beta_T, \gamma_{12}, \gamma_{21}, \gamma_{uT}, \gamma_{vT}, \alpha_T, c_0 \)

- In a complete LES all computed “on the fly” based on resolved-scale

- **Physical Interpretation of \( \gamma_{ij} \)'s Treated in** (*McDonough et al., JoT, 2003*)

- **Other Parameters to Be Discussed Here**
**DIMENSIONLESS PARAMETERS**

- Scaling Employed Leads to $Re, Pe, Gr/Re^2$

- Must Relate These to Bifurcation Parameters $\beta s$, and $\alpha$

- But Equations Solved (i.e., evaluated) on Small Local Subdomains, Im- plying Dimensionless Parameters Need Rescaling

  - $Re \rightarrow Re_h \equiv \frac{h^2||\nabla u||}{v}$, $Pe \rightarrow Pe_h \equiv \frac{h^2||\nabla u||}{v}$, $Gr \rightarrow Gr_h \equiv \frac{\beta g \delta T h^3}{v^2}$

- Rescaled parameters related to PMNS eqs. bifurcation parameters as

  $$\beta_{1,2} = 4 \left( 1 - \frac{|k|}{Re_h} \right)^2, \quad \beta_T = Pe_h, \quad \alpha_T = Gr_h / Re_h^2$$

- Desirable to Express These in Terms of Original (un-rescaled) Dimen- sionless Parameters to Permit Direct Comparisons with Experiment
Define Three Constants

\[ C_1 = \tau \left| k \right|^2, \quad C_{2,i} = \Delta u_i h_i, \quad C_3 = \left( \sum_i h_i^2 \right) \left[ \sum_{ij} (\Delta u_i / h_j)^2 \right]^{1/2} \]

Then It Can Be Shown That

\[ Re = \frac{C_1 / C_{2,i}}{1 - \beta_i / 4}, \quad \text{and} \quad Pe = \frac{C_1}{C_3 \beta_T} \]

It Can Also Be Shown That Using \( Ra = GrPr \) Leads to

\[ Ra = \frac{\alpha_T}{\tau} \frac{C_1^2}{C_{2,i} C_3 \beta_T (1 - \beta_i / 4)} \]

Finally, It Can Be Shown That

\[ Pr = \frac{1}{\beta_T} (1 - \beta_2 / 4), \quad \text{and} \quad Nu = 1 + \frac{C_1 / C_3}{\beta_T} \frac{\langle b,c \rangle}{\Delta c/h_2} \]
RESULTS

- Previously Have Shown Ability of PMNS + Thermal Energy Equation to Match Experimental Results of Gollub & Benson (*JFM*, 1980)

- Results mainly qualitative

- Time series produced by model resemble those of experiments as model parameter related to $Ra$ is increased

- But does demonstrate ability of symbol/pseudo-differential operator based models to replicate temporal physics

- Here, Additional More Quantitative Results Associated with Reproducing Experimental Correlations Are Presented
**RESULTS (Cont.)**

- Comparisons Made with Cioni *et al.* Data (*JFM*, 1997)
- High-$Ra$ Convection with $Pr = 0.025$

![Graph showing Nusselt Number vs. Rayleigh Number with two lines and data points, indicating approximate locations of bifurcations in experiments of Cioni *et al.*]
SUMMARY AND CONCLUSIONS

- Outline Given for Derivation of Discrete Dynamical System Proposed for Use as Part of LES-Like SGS Models

- Correspondence Between Model Bifurcation Parameters and Those of Original Governing Equations Demonstrated

- Possibility to Accurately Match Experimental Data Shown in Terms of Both Temporal Fluctuations and Global Quantities

- Suggests Form of Model Able to Retain Considerable Physical Detail Despite Its Simplicity and Efficiency

- Hence, Models of This Form May Be Useful as SGS Models for LES (and maybe RANS), and Also for Real-Time Control