A Compressible “Poor Man’s Navier-Stokes” Discrete Dynamical System

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Motivation for New Approach to Turbulence Modeling

- **Direct Numerical Simulation (DNS):** no modeling, permits interaction of turbulence with other physics on smallest scales. Run times for DNS $\sim O(Re^3)$.

- **Reynolds Averaged Navier stokes (RANS):** essentially all modeling, small-scale interactions impossible. Computes fairly quickly.

- **Large-Eddy Simulation (LES):** some modeling; but current models not able to treat details of SGS interactions. Run times for LES $\sim O(Re^2)$.

- Filtering of equations as done in traditional LES leads to the fundamental problem of mapping:
  
  Physics $\rightarrow$ Statistics
The New Approach — Use of Synthetic Velocity

• Approach taken by Advanced CFD Group (UK)

• Dependent variable decomposition same as LES.

\[ U(x, t) = \tilde{u}(x, t) + u^*(x, t) \]

• Mimic physics of SGS fluctuations, i.e., \( u^*, v^*, w^* \) are modeled.

• Filter solutions not equations.

• N.–S. equations now take the form,

\[ (\tilde{u} + u^*) + (\tilde{u} + u^*) \cdot \nabla(\tilde{u} + u^*) = -\nabla(\tilde{p} + p^*) + 1/Re \Delta (\tilde{u} + u^*) \]

• Directly compute \( \tilde{u} \) and \( \tilde{p} \), model \( u^* \) and \( p^* \):

\[ U = \tilde{u} + u^* \text{ and } P = \tilde{p} + p^* \]
The New Approach — Use of Synthetic Velocity (cont.)

- Form used to find fluctuating velocity components,
  \[ u^* = A M \]
  
  \( A \) – amplitude factor deduced from an extension of Kolmogorov theories.
  \( M \) – modeled employing a discrete dynamical system (DDS).

- Formulation is applied at each discrete grid point and time level.

- “The Poor man’s Navier–Stokes equations” (PMNS) in 2D.
  \[ a^{n+1} = \beta_1 a^n (1 - a^n) - \gamma_1 a^n b^n \]
  \[ b^{n+1} = \beta_2 b^n (1 - b^n) - \gamma_2 a^n b^n \]

- These DDS are derived directly from momentum equations.
Use of PMNS equations in Turbulence Modeling

2-D Incompressible Turbulent Convection:

- Experimental data taken from,
  

- Computed results from,
  
  J. M. McDonough, J. C. Holloway and M. G. Chong “A discrete dynamical system model of temporal fluctuations in turbulent convection” being prepared to be sent to J. Fluid Mech.
Derivation of 3-D Compressible PMNS equations

• Scaling of the governing equations of compressible flow gives,

\[ \rho_t + \nabla \cdot (\rho U) = 0 \]
\[ \rho \frac{DU}{Dt} = -\frac{1}{\gamma M^2} \nabla p + \frac{1}{Re} \nabla \tau_{ij} \]
\[ \rho \frac{DE}{Dt} = -\frac{1}{\gamma M^2} \nabla \cdot (pU) + \frac{1}{Pe} \nabla \cdot (k \nabla T) + \frac{1}{Re} \phi \]
\[ \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \cdot U, \quad \phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} \]

Equation of state: \( p = \rho T \)
Derivation of 3-D Compressible PMNS equations (cont.)

- Assume Fourier representations for $\rho, u, v, w, p, E$
  
  \[
  u(x,t) = \sum_{k=-\infty}^{\infty} b_k(t) \phi_k(x) , \quad v(x,t) = \sum_{k=-\infty}^{\infty} c_k(t) \phi_k(x) , \quad w(x,t) = \sum_{k=-\infty}^{\infty} g_k(t) \phi_k(x)
  \]

  \[
  \rho(x,t) = \sum_{k=-\infty}^{\infty} a_k(t) \phi_k(x) , \quad P(x,t) = \sum_{k=-\infty}^{\infty} e_k(t) \phi_k(x) , \quad E(x,t) = \sum_{k=-\infty}^{\infty} h_k(t) \phi_k(x)
  \]

  assume $\phi_k$ to be orthonormal and in $C_0^\infty$

- Apply Galerkin procedure
  
  - Substitute solution representations into governing equations.
  
  - Commute differentiation and summation.
  
  - Calculate Galerkin inner products.
  
  - Reduce to a single wave vector.
  
  - Construct forward Euler discretisation, rearrange to define bifurcation parameters to get PMNS equations.
3-D Compressible PMNS equations

- PMNS equations for 3-D compressible flow:

\[
a^{n+1} = a^n - \gamma_{11}a^n b^n - \gamma_{12}a^n c^n - \gamma_{13}a^n g^n
\]

\[
b^{n+1} = \beta_1 b^n (1 - b^n) - \gamma_{21}b^n c^n - \gamma_{22}b^n g^n + a b^n + \frac{1}{a^n} [a b^n - \eta_1 e^n] + \frac{1}{a^n} \left[ \frac{1}{3} \chi_1 b^n + \frac{1}{3} \xi_1 c^n + \frac{1}{3} \xi_2 g^n \right]
\]

\[
c^{n+1} = \beta_2 c^n (1 - c^n) - \gamma_{31}b^n c^n - \gamma_{32}b^n g^n + a c^n + \frac{1}{a^n} [a c^n - \eta_2 e^n] + \frac{1}{a^n} \left[ \frac{1}{3} \chi_2 c^n + \frac{1}{3} \xi_2 b^n + \frac{1}{3} \xi_3 g^n \right]
\]

\[
g^{n+1} = \beta_3 g^n (1 - g^n) - \gamma_{41}b^n g^n - \gamma_{42}b^n c^n + a g^n + \frac{1}{a^n} [a g^n - \eta_3 e^n] + \frac{1}{a^n} \left[ \frac{1}{3} \chi_3 g^n + \frac{1}{3} \xi_3 b^n + \frac{1}{3} \xi_3 c^n \right]
\]

\[
h^{n+1} = h^n - \gamma_{51} h^n b^n - \gamma_{52} h^n c^n - \gamma_{53} h^n g^n - \frac{1}{a^n} \left[ \eta_1 e^n b^n + \eta_2 e^n c^n + \eta_3 e^n g^n \right] + \frac{1}{a^n} \kappa T^n
\]

\[
+ \frac{1}{a^n} \xi_1 \left[ \frac{4}{3} (b^n)^2 + (c^n)^2 + (g^n)^2 \right] + \frac{1}{a^n} \xi_2 \left[ \frac{4}{3} (c^n)^2 + (b^n)^2 + (g^n)^2 \right] + \frac{1}{a^n} \xi_3 \left[ \frac{4}{3} (g^n)^2 + (b^n)^2 + (c^n)^2 \right]
\]

\[
+ \frac{1}{a^n} \left[ \lambda_1 \left( \frac{2}{3} b^n c^n \right) + \lambda_2 \left( \frac{2}{3} c^n g^n \right) + \lambda_1 \left( \frac{2}{3} b^n g^n \right) \right]
\]
Bifurcation Parameters

• 3–D compressible PMNS equations has thirty six bifurcation parameters.

• Bifurcation parameters can be grouped into three families corresponding to whether they depend on $Re$, $M$ or strain rates.

• Is there a closure problem?

• All bifurcation parameters are calculated using the resolved part of LES and hence known before hand.
Time Series and Power Spectral Density

Periodic
Subharmonic
Phase lock
Quasi-periodic
Noisy-Subharmonic
Noisy-Phase lock
Noisy-Quasi-Periodic with fund.
Noisy-Quasi-Periodic without fund.
Broadband-with fund.
Broadband-with dif. fund.
Broadband-without fund.
Regime Maps (Bifurcation Diagrams)

\[ \beta_1 = 4 \left(1 - \frac{\tau z |k|^2}{Re}\right) \]

\[ \gamma_{11} = \tau B_{lm,k}^1 \]

\[ \psi_1 = \frac{\tau k_1}{\gamma M^2} \]
Summary

• Shortcomings of present day turbulence models was presented.

• Use of synthetic velocity in turbulence modeling was presented.

• Derivation of 3-D compressible PMNS was presented.

• 3-D compressible PMNS produces all non-trivial types of behavior as observed for the incompressible case.