1. Consider the following discrete-time multi-input state variable model:

$$x_{k+1} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} x_k + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} w_k \; 0$$

a) Find the eigenvalues of the open-loop system. Solution: the eigenvalues are {-2, -2}

b) Is the system stable? Solution: No! Both eigenvalues are outside the Unit Circle.

c) Is the system controllable? Solution: $M = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -6 & 4 \\ -1 & 1 & 2 & -2 \end{bmatrix}$ which has rank=2. Thus, the system is completely controllable.

c) If yes, design a feedback regulator law, $w_k = -kx_k$ such that the closed-loop system has eigenvalues {.5 .25}. Solution: Since the system has repeated eigenvalues, we must use $w_k = -K_F x_k + \hat{w}_k$ where K_F is a random mxn matrix. If we let $K_F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then the new next-state equation is $x_{k+1} = (\hat{A} - \hat{B}K_F)x_k + \hat{B}\hat{w}_k = A_{new}x_k + \hat{B}\hat{w}_k = \begin{bmatrix} -4 & 2 \\ 1 & -2 \end{bmatrix} x_k + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} w_k$. The eigenvalues of A_{new} are {-4.73, -1.27} which are distinct. Now, we may reduce the system to an equivalent single input system using the assignment, $\hat{w}_k = k_s w_{s_k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_{s_k}$. With this assignment, the next state equation is $x_{k+1} = \begin{bmatrix} -4 & 2 \\ 1 & -2 \end{bmatrix} x_k + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} w_{s_k} = \begin{bmatrix} -4 & 2 \\ 1 & -2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_{s_k} = A_{new}x_k + b_s w_{s_k}$. We need to check that this single input system is still controllable: $M_s = \begin{bmatrix} b_s & A_{new}b_s \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ which has rank=2. Thus, the single input system is controllable and we can transform it into phase-variable form. The characteristic equation is $s^2+6s+6=0$. Thus, $A_{pv} = \begin{bmatrix} 0 & 1 \\ -6 & -6 \end{bmatrix}, and\; b_{pv} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Using the similarity transformation, $x_k=T_{pv}z_{pvc}$, and setting $w_{sk} = -k_{pv}z_{pvk}$, we obtain the following equation: $z_{pv_{k+1}} = \begin{bmatrix} 0 & 1 \\ -6 & -6 \end{bmatrix} z_{pv_k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_{s_k} = \begin{bmatrix} 0 & 1 \\ -(6+k_{pv_0}) & -(6-k_{pv_1}) \end{bmatrix} z_{pv_k}$. This system has the characteristic equation, $s^2+(6+kpv_1)s+(6+kpv_0)=0$. The desired characteristic equation is $(s-0.5)(s-0.25)= s^2-0.75s+0.125=0$. Equating the two, we see that $kpv_0=-5.875$ and $kpv_1=-6.75$. The entire control is given by $w_k=-(K_F+k_sk_{pv}T_{pv}^{-1})x_k$ or $w_k = -\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} -5.875 & -6.75 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\right) x_k = -\begin{bmatrix} -5.75 & 7.625 \\ -6.75 & 7.625 \end{bmatrix} x_k$. If we check the eigenvalues of (A-BK), we find that the closed-loop eigenvalues are {0.5 0.25}. **Note: This answer is not unique and depends upon the choice of K_F and k_s. Your answers may vary depending upon how you drive, but check the eigenvalues of (A-BK) to verify that your solution is correct.**

d) What is the settling time of the closed-loop system if T_s = 10msec? Solution: The closed-loop eigenvalue closest to the Unit Circle is 0.5. Hence, the settling time is t_s= $-4T_s/\ln(0.5)$ = 57.7 msec.

e) If $x_{k+1} = \hat{A}x_k + \hat{B}w_k \; 0$ determine which systems are stabilizable:

i) $\hat{A} = \begin{bmatrix} -1 & -3 \\ 3/2 & 7/2 \end{bmatrix}$ $\hat{B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ Solution: The controllability matrix for this system is $M = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ which has rank=1. The easiest way to determine which eigenvalues are controllable and which are not is to put the system into controllable canonical form by using the similarity transformation, $x_k = T_{ccf} z_{ccfk}$. For the given system, $T_{ccf} = \begin{bmatrix} r\ lin.ind. & n-r\ more \\ cols.of\ M & lin.ind.cols. \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Applying this transformation, we find $T_{ccf}^{-1}\hat{A}T_{ccf} = \begin{bmatrix} 2 & 1.5 \\ 0 & 1/2 \end{bmatrix}$, and $T_{ccf}^{-1}\hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and we see that the uncontrollable eigenvalue is ½ which is stable. Thus, the system is stabilizable!

0ii) $\hat{A} = \begin{bmatrix} 11 & 10 \\ -5 & -4 \end{bmatrix}$ $\hat{B} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ Solution: The controllability matrix for this system is $M = \begin{bmatrix} -1 & -2 & -1 & -2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ which has rank=1. Again, the easiest way to determine which eigenvalues are controllable and which are not is to put the system into controllable canonical form by using the similarity transformation, $x_k = T_{ccf} z_{ccfk}$. For the given system, $T_{ccf} = \begin{bmatrix} r\ lin.ind. & n-r\ more \\ cols.of\ M & lin.ind.cols. \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Applying this transformation, we find $T_{ccf}^{-1}\hat{A}T_{ccf} = \begin{bmatrix} 1 & -5 \\ 0 & 6 \end{bmatrix}$, and $T_{ccf}^{-1}\hat{B} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and we see that the uncontrollable eigenvalue is 6 which is unstable. Thus, the system is **NOT** stabilizable!

0 iii) $\hat{A} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$ $\hat{B} = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$ Solution: The controllability matrix for this system is $M = \begin{bmatrix} -1 & 4 & -2 & 4 \\ 1 & -2 & 2 & -2 \end{bmatrix}$ which has rank=2. Thus, the system is completely controllable0 and, of course, stabilizable.

2. Consider the following discrete-time state variable model:

$$x_{k+1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} w_k \qquad x_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} 0$$

a) Find the eigenvalues of the open-loop system. Solution: the eigenvalues are {4,0.5, 2}.

b) Is the system stable? Solution: No! both 2 and 4 are outside of the Unit Circle

c) Is the system controllable? Is it stabilizable? Solution: Since the system is decoupled with distinct eigenvalues, we can see that the controllable eigenvalues are {4,2} while the uncontrollable eigenvalue is ½. Obviously, the system is **NOT** completely controllable but since the uncontrollable eigenvalue is stable, the system is stabilizable.

d) If yes, design a feedback regulator law, $w_k = -kx_k$ such that all the controllable eigenvalues are {1/4}. Solution: we must first put the system into controllable canonical form in order to isolate the controllable modes from the uncontrollable modes. To do this, we must first find $M = \begin{bmatrix} 2 & 0 & 8 & 0 & 32 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -4 & 0 & -8 \end{bmatrix}$ which has rank=2 as expected. For the given system, $T_{ccf} = \begin{bmatrix} r\ lin.ind. & n-r\ more \\ cols.of\ M & lin.ind.cols. \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$. Applying this transformation, $x_k=T_{ccf}z_{ccfk}$, we find that

$$z_{ccf_{k+1}} = T_{ccf}^{-1}\hat{A}T_{ccf}z_{ccf_k} + T_{ccf}^{-1}\hat{B}w_k = \begin{bmatrix} A_c & JUNK \\ 0 & A_{\bar{c}} \end{bmatrix} z_{ccf_k} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} w_k = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} z_{ccf_k} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} w_k .$$

Note that if we let $w_k=-[K_c \mid 0]z_{ccfk}=-[K_c \mid 0]T_{ccf}^{-1}x_k$, then the problem has been reduced to finding Kc which will set the eigenvalues of $(A_c-B_cK_c)$ where $A_c = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Choosing $k_s = [1\ 1]^T$, we find the single input system is defined by $A_c = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $b_s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In phase variable form, we have $A_{pv} = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}$ *and* $b_{pv} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ where $T_{pv} = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ -4 & 1 \end{bmatrix}$. One can easily verify that the appropriate k_{pv} which will set the eigenvalues of $(A_{pv}-b_{pv}k_{pv})$ to {¼, ¼} is k_{pv} = [-7.9375 5.5000]. Hence, $K_c = k_s k_{pv} T_{pv}^{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} -7.9375 & 5.5 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7.0312 & -1.5312 \\ 7.0312 & -1.5312 \end{bmatrix}$ and the final control is $w_k=-[K_c \mid 0]z_{ccfk}=-[K_c \mid 0]T_{ccf}^{-1}x_k = -\begin{bmatrix} 7.0312 & -1.5312 & 0 \\ 7.0312 & -1.5312 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}^{-1} x_k = -\begin{bmatrix} 3.5156 & 0 & 0.7656 \\ 3.5156 & 0 & 0.7656 \end{bmatrix} x_k$. We can check this solution by finding the eigenvalues of (A-BK) and we obtain {0.5, 0.25, 0.25}. Note that the uncontrollable eigenvalue, 0.5, does not vary (i.e., is invariant under feedback) as expected.

e) What is the settling time of the closed-loop system if T_s = 10msec? Solution: The closed-loop eigenvalue closest to the Unit Circle is 0.5. Hence, the settling time is $t_s= -4T_s/\ln(0.5)$ = 57.7 msec.