EE572 - Solution to HW #8

1. a) Solution: Consider the following discrete-time state variable model (don't use Matlab):

$$x_{k+1} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_k \quad 1$$

The characteristic equation is s^2+-8s+0=0. Thus, $\hat{A}_{pv} = \begin{bmatrix} 0 & 1 \\ 0 & 8 \end{bmatrix} and\ \hat{b}_{pv} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, the similarity transformation we are looking for is given by $T_{pv}=MM_{pv}^{-1}$ or $T_{pv} = \begin{bmatrix} 1 & 12 \\ 2 & 12 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 12 \\ 2 & 12 \end{bmatrix}\begin{bmatrix} -8 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -4 & 2 \end{bmatrix}$

b) Now find values for A_{pv} and B_{pv} where $z_{pv}((k+1)T_s)= A_{pv}\ z_{pv}(kT_s)+ B_{pv}\ w(kT_s)$ Solution: $\hat{A}_{pv} = \begin{bmatrix} 0 & 1 \\ 0 & 8 \end{bmatrix} and\ \hat{b}_{pv} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c) Design a feedback control law, $w_k = -\underline{k}x_k$ such that the closed-loop eigenvalues are {0.4 0.5}. Solution: In phase variable form, we have the following next state equation: $z_{pv_{k+1}} = \begin{bmatrix} 0 & 1 \\ 0 & 8 \end{bmatrix} z_{pv_k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$. By letting $w_k = -\underline{kz_{pv}} = -[k_0\ k_1]z_{pv}$, we obtain $z_{pv_{k+1}} = \begin{bmatrix} 0 & 1 \\ -k_0 & -(k_1-8) \end{bmatrix} z_{pv_k}$ which has the characteristic equation $s^2 + (k_1-8)s + k_0 = 0$. Equating this to our desired characteristic equation of $(s-0.4)(s-0.5) = s^2 - 0.9s + 0.2 = 0$, we find that k_0 =0.2 and k_1=7.1. Thus, $w_k = -\underline{k}z_{pv} = -[0.2\ 7.1]z_{pv} = -[0.2\ 7.1]T_{pv}^{-1}x_k = -[\ 2.4\ \ 2.35]x_k$

d) What is settling time if T_s = 10 msec? Solution: The closed-loop eigenvalues are 0.4 and 0.5. Thus, 0.5 is the eigenvalue closest to the unit circle. Consequently, the settling time is $t_s = -4 \times T_s/\ln(0.5)$ = 57.7 msec.

2. Consider the following discrete-time state variable multi-input model (use Matlab):

$$x_{k+1} = \begin{bmatrix} 3 & 0.5 & 1 \\ -1 & 0.75 & -0.5 \\ 0.5 & 0.625 & 2.25 \end{bmatrix} x_k + \begin{bmatrix} 0.5 & 0 \\ -0.25 & 1 \\ 0.625 & 0 \end{bmatrix} w_k \quad 2$$

a) Determine if the system is completely controllable. Solution: The controllability matrix is $M=[B\ AB\ A^2B]$ which has rank = 3. Therefore, our system is completely controllable!

b) Find a similarity transformation, $x(kT_S)=T_{pv}z_{pv}(KT_s)$ such that z_{pv} is in phase variable form. Solution: First, we must reduce the system into a single-input controllable system by making the assignment, $w_k = \underline{k_s}w_s$ where $\underline{k_s}$ is an mx1 random vector. Let $\underline{k_s} = [1\ 1]^T$. This assignment results in the following single-input system:

$$x_{k+1} = \begin{bmatrix} 3 & 0.5 & 1 \\ -1 & 0.75 & -0.5 \\ 0.5 & 0.625 & 2.25 \end{bmatrix} x_k + \begin{bmatrix} 0.5 \\ 0.75 \\ 0.625 \end{bmatrix} w_{s_k}$$

The rank of the controllability matrix for this system is three. Thus, we now have a controllable single-input equivalent system and can proceed to transform it into phase variable form. From the Matlab output, we see that $T_{pv} = M_s M_{pv}^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 6 & -19/4 & 3/4 \\ 0 & -13/8 & 5/8 \end{bmatrix}$

c) Now find values for A_{pv} and B_{pv} where $z_{pv}((k+1)T_s) = A_{pv}\, z_{pv}(kT_s) + B_{pv}\, w(kT_s))$. Solution: The characteristic polynomial of A is $s^3 - 6s^2 + 11s - 6 = 0$. Therefore, $A_{pv} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$ *and* $b_{pv} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

d) Design a feedback control law, $w_k = -\underline{k}x_k$ such that the closed-loop eigenvalues are {0.5 0.6 0.7}. Solution: By letting $w_s = -\underline{k}\underline{z_{pv}} = -[k_0\ k_1\ k_2]z_{pv}$, we obtain $z_{pv_{k+1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_0-6) & -(k_1+11) & -(k_2-6) \end{bmatrix} z_{pv_k}$ which has the characteristic equation $s^3 + (k_2-6)s^2 + (k_1+11)s + (k_0-6) = 0$. Equating this to our desired characteristic equation of $(s-0.5)(s-0.6)(s-0.7) = s^3 - 1.8s^2 + 1.07s - 0.21 = 0$, we find that $k_0 = 5.79$ and $k_1 = -9.93$ and $k_2 = 4.2$. Thus, $w_s = -\underline{k}z_{pv} = -[4.2\ -9.93\ 5.79]z_{pv} = -[4.2\ -9.93\ 5.79]T_{pv}^{-1}x_k = -[\,4.615\quad 0.965\quad 1.87]x_k$ and the final, multi-input control is found from $w_k = \underline{k_s}w_s$

$$= -\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 4.615 & 0.965 & 1.87 \end{bmatrix} x_k = -\begin{bmatrix} 4.615 & 0.965 & 1.87 \\ 4.615 & 0.965 & 1.87 \end{bmatrix} x_k$$

e) What is settling time if T_s = 10 msec? Solution: The closed-loop eigenvalues are 0.5, 0.6, and 0.7. Thus, 0.7 is the eigenvalue closest to the unit circle. Consequently, the settling time is $t_s = -4 \times T_s/\ln(0.7) = 112.1$ msec.

APPENDIX:

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» %Matlab diary for problem 2
» a=[3 .5 1;-1 0.75 -0.5;0.5 0.625 2.25]
a =
    3.0000    0.5000    1.0000
   -1.0000    0.7500   -0.5000
    0.5000    0.6250    2.2500
» b=[0.5 0;-0.25 1;0.625 0]
b =
    0.5000         0
   -0.2500    1.0000
    0.6250         0
» %Form controllability matrix, m
» m=[b a*b a^2*b]
m =
    0.5000         0    2.0000    0.5000    7.0000    2.5000
   -0.2500    1.0000   -1.0000    0.7500   -3.5000   -0.2500
    0.6250         0    1.5000    0.6250    3.7500    2.1250
» rank(m)
ans =
     3
» %Form single-input system
» ks=[1;1]
ks =
     1
     1

» bs=b*ks
bs =
    0.5000
    0.7500
    0.6250
» %Check that single-input system is still controllable
» ms=[bs a*bs a^2*bs]
ms =
    0.5000    2.5000    9.5000
    0.7500   -0.2500   -3.7500
    0.6250    2.1250    5.8750
» rank(ms)
ans =
     3
» poly(a)
```

```
ans =
    1.0000   -6.0000   11.0000   -6.0000
» %Find phase-variable form:
» apv=[0 1 0;0 0 1;6 -11 6]
apv =
     0     1     0
     0     0     1
     6   -11     6
» bpv=[0;0;1]
bpv =
     0
     0
     1
» mpv=[bpv apv*bpv apv^2*bpv]
mpv =
     0     0     1
     0     1     6
     1     6    25
» Tpv=ms*inv(mpv)
Tpv =
         0   -0.5000    0.5000
    6.0000   -4.7500    0.7500
         0   -1.6250    0.6250
» poly([0.5 0.6 0.7])
ans =
    1.0000   -1.8000    1.0700   -0.2100
» k0=ans(4)+apv(3,1)
k0 =
    5.7900
» k1=ans(3)+apv(3,2)
k1 =
   -9.9300
» k2=ans(2)+apv(3,3)
k2 =
    4.2000
» [k0 k1 k2]*inv(Tpv)
ans =
    4.6150    0.9650    1.8700
»k= ks*ans
k =
    4.6150    0.9650    1.8700
    4.6150    0.9650    1.8700
» %Check closed-loop eigenvalues:
» eig(a-b*k)
ans =
    0.6000
    0.7000
    0.5000
» ts=-4*0.01/log(0.7)
ts =
    0.1121
```