

EE572 - Solution to HW #7

1. a) i) **Solution:** Find eigenvalues from $\det[sI-A] = 0 = (s-3)(s-1)$. Therefore, eigenvalues are $s_1=3$ and $s_2=1$. The eigenvectors are found from $[s_i I-A]P_i=0$. The eigenvectors are $P_1=[-1 \ 1]^T$ and $P_2=[-1 \ -1]^T$. Thus, a similarity transformation which will decouple the given system is to let $x_k=Pz_k$. In doing so, we obtain:

$$z_{k+1} = P^{-1}\hat{A}Pz_k + P^{-1}\hat{B}w_k = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} z_k + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k.$$

Clearly, the controllable eigenvalue is $s_2=1$ and the uncontrollable eigenvalue is $s_1=3$.

- ii) **Solution:** Find eigenvalues from $\det[sI-A] = 0 = (s-1/2)(s+3)(s+4)$. Therefore, eigenvalues are $s_1=1/2$, $s_2=-3$ and $s_3=-4$. The eigenvectors are found from $[s_i I-A]P_i=0$. The eigenvectors are $P_1=[1 \ 0 \ 0]^T$, $P_2=[1 \ 7/4 \ 0]^T$, and $P_3=[0 \ 0 \ 1]^T$. Thus, a similarity transformation which will decouple the given system is to let $x_k=Pz_k$.

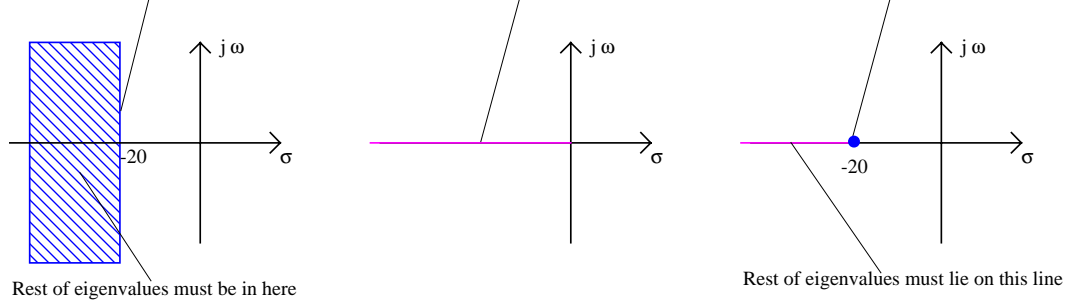
$$z_{k+1} = P^{-1}\hat{A}Pz_k + P^{-1}\hat{B}w_k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 7/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 & -2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 7/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} z_k + \begin{bmatrix} 1 & 1 & 0 \\ 0 & 7/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} w_k$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} z_k + \begin{bmatrix} 3/7 \\ 4/7 \\ 0 \end{bmatrix} w_k$$

- c) **Solution:** For problem 1a) part i), $\{3\}$ is an unstable eigenvalue and $\{1\}$ is marginally stable. For problem 1a) part ii), $\{-3,-4\}$ are unstable eigenvalues and $\{1/2\}$ is stable.

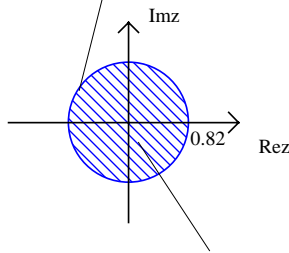
- d) Find the region in the s-plane which corresponds to the following design specifications:

A settling time (2%) of 0.2 sec and no overshoot (i.e., no oscillation). **Solution:** $\text{Re}[s_i]_{\max} = -4/t_s = -20$.
 To meet settling time, at least one eigenvalue must lie on this line
 To meet zero overshoot, all eigenvalues must lie on this line
 The intersection of these two criteria, at least one eigenvalue must lie at this point



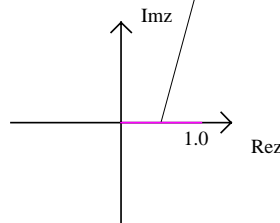
- e) Repeat part d) in the Z-plane if $T_s = 10$ msec. **Solution:** To map this region into the Z-plane, we must use the relationship, $z = e^{sT_s}$. Note that $e^{-20T_s} = 0.82$ for $T_s = 10$ msec.

To meet settling time, at least one eigenvalue must lie on this circle

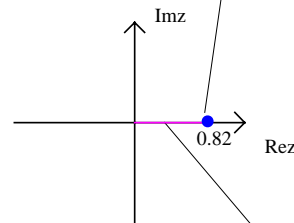


f) Rest of eigenvalues must be in here

To meet zero overshoot, all eigenvalues must lie on this line



The intersection of these two criteria, at least one eigenvalue must lie at this point



Rest of eigenvalues must lie on this line

2. Consider the following discrete-time state variable model:

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_k$$

a) Is the system completely controllable? Solution: From MATLAB, we can easily find the similarity transformation to decouple the system. By doing so, we obtain the following

$$z_{k+1} = \begin{bmatrix} -0.38 + j0.97 & 0 & 0 \\ 0 & -0.38 - j0.97 & 0 \\ 0 & 0 & 2.76 \end{bmatrix} z_k + \begin{bmatrix} 0.2 + j0.18 \\ 0.2 - j0.18 \\ 0.76 \end{bmatrix} w_k$$

Obviously, the system is completely controllable. Also, note that the original system is in phase variable form. Because we can completely control all eigenvalues of a system in phase variable form, **EVERY SYSTEM IN PHASE VARIABLE FORM IS CONTROLLABLE!**

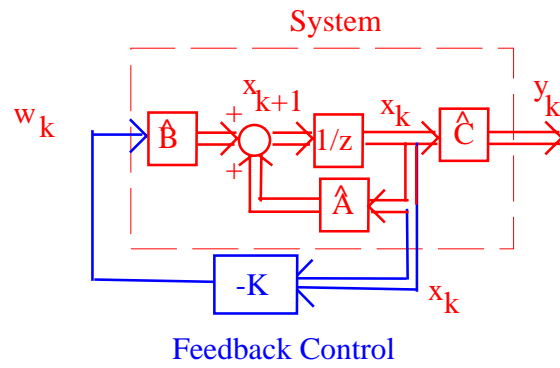
b) Is the system stable? Solution: No. The magnitude of all three eigenvalues are outside the U.C. ($|-0.38 + j0.97| = 1.04$). If **any** eigenvalues are outside the U.C., the system is unstable

c) If the system is completely controllable (which it is), design a feedback control law, $w_k = -\underline{k}x_k$ such that the closed-loop eigenvalues are $\{.4, .4, .4\}$. Solution: If we let $w_k = -\underline{k}x_k = -[k_0 \ k_1 \ k_2]x_k$ then we obtain:

$$x_{k+1} = (A_{pv} - b_{pv}k)x_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_0 - 3) & -(k_1 - 1) & -(k_2 - 2) \end{bmatrix} x_k$$

Because this system is still in the special “phase variable” form, we know the characteristic equation is: $s^3 + (k_2 - 2)s^2 + (k_1 - 1)s + (k_0 - 3) = 0$. Our desired characteristic equation is $(s - 0.4)^3 = s^3 - 1.2s^2 + 0.48s - 0.064 = 0$. Equating coefficients, we find that $k_2 = 0.8$, $k_1 = 1.48$, $k_0 = 2.936$. Or $w_k = -[2.936 \ 1.48 \ 0.8]x_k$

d) Complete the block diagram of your new closed-loop system. Solution:



- e) What is settling time of the new closed-loop system if $T_s = 10$ msec? Solution: $0.4 = e^{-4T_s/t_s}$ or $t_s = -4T_s / \ln(0.4) = 43.7$ msec