EE572 - Solution to HW #6

1. a) Use the fact that the solution to, $Z\{x_{k+1} = \hat{A}x_k\} = zX(z) - zx_0 = \hat{A}X(z)$ *or* $X(z) = [zI - \hat{A}]^{-1} zx_0$ to find the Z-transform of $\hat{A}^k$ (hint: take the Z-transform of both $x_{k+1} = \hat{A}x_k$ *and* $x_k = \hat{A}^k x_0$ and solve for the Z-transform of $\hat{A}^k$)

Solution:

Taking the z-transform we see that: $Z\{x_{k+1} = \hat{A}x_k\} = zX(z) - zx_0 = \hat{A}X(z)$ *or* $X(z) = [zI - \hat{A}]^{-1} zx_0$. We can also take the z-transform of the solution: $Z\{x_k = \hat{A}^k x_0\} = X(z) = Z\{\hat{A}^k\} x_0$. Comparing both answers for X(z) and eliminating x_0 from both these expressions, we find that $Z\{\hat{A}^k\} = [zI - \hat{A}]^{-1} z$ *or* $\hat{A}^k = Z^{-1}\{[zI - \hat{A}]^{-1} z\}$

b) Use your answer to part c) and the Z-transform convolution theorem given in class to show that the solution to the discrete state variable model, $Z\{x_{k+1} = \hat{A}x_k + \hat{B}w_k\} = zX(z) - zx_0 = \hat{A}X(z) + \hat{B}W(z)$ is

$$x_k = \hat{A}^k x_0 + \sum_{j=-\infty}^{\infty} \hat{A}^{k\text{-}j} g_j \quad or \quad x_k = \hat{A}^k x_0 + \sum_{j=-\infty}^{\infty} \hat{A}^{k\text{-}j} \hat{B} w_{j-1}$$

Solution:

Taking the z-transform of both sides, we obtain: $Z\{x_{k+1} = \hat{A}x_k + \hat{B}w_k\} = zX(z) - zx_0 = \hat{A}X(z) + \hat{B}W(z)$ or $X(z) = [zI - \hat{A}]^{-1} zx_0 + [zI - \hat{A}]^{-1} \hat{B}W(z) = [zI - \hat{A}]^{-1} zx_0 + [zI - \hat{A}]^{-1} z\hat{B}z^{-1}W(z)$. If we replace $\hat{B}z^{-1}W(z)$ *by* $G(z)$ (i.e., $g_k = \hat{B}w_{k-1}$) and take the inverse Z-transform using the results of part c) and the convolution theorem, we obtain $x_k = \hat{A}^k x_0 + \sum_{j=-\infty}^{\infty} \hat{A}^{k\text{-}j} g_j$ *or* $x_k = \hat{A}^k x_0 + \sum_{j=-\infty}^{\infty} \hat{A}^{k\text{-}j} \hat{B} w_{j-1}$. If we assume that the input w_k is zero before k=0 and that the system is causal, we find $x_k = \hat{A}^k x_0 + \sum_{j=1}^{k} \hat{A}^{k\text{-}j} \hat{B} w_{j-1}$, which is the desired result!

2. Given the continuous state variable model, $\dot{x} = Ax + Bw$, and the corresponding discrete next-state approximation to this model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$ with sampling period T. Here are some neat facts that are not too difficult to prove (but, I won't ask you to do so):

If we use the approximation that $\hat{A} = e^{AT}$, then

i) The eigenvalues of $\hat{A}$ are $e^{s_i T}$ where s_i are the eigenvalues of our original A matrix
ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix
iii) If A is stable, then our approximation, $\hat{A}$ is also stable for all possible choices of the sampling period, T.

If, however, we use the approximation that $\hat{A} = I + TA$, then

i) The eigenvalues of $\hat{A}$ are $1+s_iT$ where s_i are the eigenvalues of our original A matrix
ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix
iii) If A is stable, then our approximation, $\hat{A}$ might be UNSTABLE for certain possible choices of the sampling period, T.

2a) Start with the continuous time system, $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_k, x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Find the eigenvectors and eigenvalues of the A matrix then determine the stability of the system.

Solution:

Find eigenvalues from det[sI-A] = 0 = (s+1)(s+2). Therefore, eigenvalues are s_1=-1 and s_2=-2. The eigenvectors are found from $[s_iI-A]P_i=0$. The eigenvectors are $P_1=[1\ -1]^T$ and $P_2=[1\ -2]^T$. Thus, the continuous time system is asymptotically stable.

b) Now, find the discrete, next-state model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$, using the approximation that $\hat{A} = e^{AT}$ and the sampling period, T = ln(2) seconds. Find the eigenvectors and eigenvalues of the $\hat{A}$ matrix then determine the stability of the system. Verify that:

i) The eigenvalues of $\hat{A}$ are $e^{s_i T}$ where s_i are the eigenvalues of our original A matrix

ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix

iii) If A is stable, then our approximation, $\hat{A}$ is also stable for all possible choices of the sampling period, T.

Solution:

$$e^{At} = Pe^{St}P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

To find values for $\hat{A}, and\ \hat{B}$ we set:

$$\hat{A} = e^{AT_s} = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ -2e^{-T} + 2e^{-2T} & -e^{-T} + 2e^{-2T} \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \\ -1/2 & 0 \end{bmatrix}, and$$

$$\hat{B} = \int_0^{T_s} e^{At}dtB = A^{-1}[e^{AT_s} - I]B = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 3/4-1 & 1/4 \\ -1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/4 \end{bmatrix}$$

The eigenvalues of $\hat{A}$ are {1/2 , ¼} which are the same as $e^{s_i T} = \{e^{-1\ln(2)}, e^{-2\ln(2)}\}$ = {1/2 , ¼}

The eigenvectors of $\hat{A}$ are $P_1=[1\ -1]^T$ and $P_2=[1\ -2]^T$ which are indeed the same as the eigenvectors of our original A matrix.

Finally, the eigenvalues of $\hat{A}$ are $e^{s_i T} = \{e^{-1T_s}, e^{-2T_s}\}$ which will be inside the unit circle (i.e., stable) for all positive T_s values.

c) Finally, find the discrete, next-state model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$, using the approximation that $\hat{A} = I + TA$ and the sampling period, T = 2 seconds. Find the eigenvectors and eigenvalues of the $\hat{A}$ matrix then determine the stability of the system. Verify that:

i) The eigenvalues of $\hat{A}$ are $1+s_iT$ where s_i are the eigenvalues of our original A matrix

ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix

iii) If A is stable, then our approximation, $\hat{A}$ might be UNSTABLE for certain possible choices of the sampling period, T.

Solution:

$$\hat{A} = I + T_sA = \begin{bmatrix} 1+0 & 2 \\ -4 & 1-6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}, \hat{B} = T_sB = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Now, the eigenvalues of $\hat{A}$ are {-1, -3} which are the same as $1+s_iT$ = {1-2, 1-4} = {-1, -3}.

The eigenvectors of $\hat{A}$ are $P_1=[1\ -1]^T$ and $P_2=[1\ -2]^T$ which are indeed the same as the eigenvectors of our original A matrix.

Note that while our original A matrix was stable, using this approximation method we obtain a discrete model that is UNSTABLE (i.e., has an eigenvalue outside of the unit circle). Thus, if we use this method, we must be extra careful when we choose our sampling time, T_s.