1\. a) Use the fact that the solution to, $x_{k+1} = \hat{A}x_k$ *is* $x_k = \hat{A}^k x_0$ to find the Z-transform of $\hat{A}^k$ (hint: take the Z-transform of both $x_{k+1} = \hat{A}x_k$ *and* $x_k = \hat{A}^k x_0$ and solve for the Z-transform of $\hat{A}^k$)

b) Use your answer to part c) and the Z-transform convolution theorem given in class to show that the solution to the discrete state variable model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$ is $x_k = \hat{A}^k x_0 + \sum_{j=1}^{k} \hat{A}^{k-j} B w_{j-1}$

2\. Given the continuous state variable model, $\dot{x} = Ax + Bw$, and the corresponding discrete next-state approximation to this model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$ with sampling period T. Here are some neat facts that are not too difficult to prove (but, I won't ask you to do so):

If we use the approximation that $\hat{A} = e^{AT}$, then

- i) The eigenvalues of $\hat{A}$ are $e^{s_i T}$ where s_i are the eigenvalues of our original A matrix
- ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix
- iii) If A is stable, then our approximation, $\hat{A}$ is also stable for all possible choices of the sampling period, T.

If, however, we use the approximation that $\hat{A} = I + TA$, then

- i) The eigenvalues of $\hat{A}$ are $1+s_i T$ where s_i are the eigenvalues of our original A matrix
- ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix
- iii) If A is stable, then our approximation, $\hat{A}$ might be UNSTABLE for certain possible choices of the sampling period, T.

2a) Start with the continuous time system, $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w_k, x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Find the eigenvectors and eigenvalues of the A matrix then determine the stability of the system.

b) Now, find the discrete, next-state model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$, using the approximation that $\hat{A} = e^{AT}$ and the sampling period, T = ln(2) seconds. Find the eigenvectors and eigenvalues of the $\hat{A}$ matrix then determine the stability of the system. Verify that:

- i) The eigenvalues of $\hat{A}$ are $e^{s_i T}$ where s_i are the eigenvalues of our original A matrix
- ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix
- iii) If A is stable, then our approximation, $\hat{A}$ is also stable for all possible choices of the sampling period, T.

c) Finally, find the discrete, next-state model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k$, using the approximation that $\hat{A} = I + TA$ and the sampling period, T = 2 seconds. Find the eigenvectors and eigenvalues of the $\hat{A}$ matrix then determine the stability of the system. Verify that:

- i) The eigenvalues of $\hat{A}$ are $1+s_i T$ where s_i are the eigenvalues of our original A matrix
- ii) The eigenvectors of $\hat{A}$ are the same as the eigenvectors of our original A matrix
- iii) If A is stable, then our approximation, $\hat{A}$ might be UNSTABLE for certain possible choices of the sampling period, T.