EE572

Sol'n to RISHW #5*

Given the continuous state variable model, $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} w,\ x(0) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, and $y = \begin{bmatrix} 3 & 4 \end{bmatrix} x$

the corresponding discrete next-state approximation to this model, $x_{k+1} = \hat{A}x_k + \hat{B}w_k,\ x_0$, $y_k = \hat{C}x_k$ with sampling period T.

a) Calculate the zero input response for t=1,2, and 3 seconds (hint: $x_{zero\text{-}input}(t) = e^{At}x(0)$)

Sol'n: $x_{zero-input} = e^{At}x(0) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5e^{-t} \\ 6e^{-3t} \end{bmatrix}$

For t=1,2, and 3 seconds: $x_{zero-input}(1) = \begin{bmatrix} 5e^{-1} \\ 6e^{-3} \end{bmatrix}, x_{zero-input}(2) = \begin{bmatrix} 5e^{-2} \\ 6e^{-6} \end{bmatrix}, x_{zero-input}(3) = \begin{bmatrix} 5e^{-3} \\ 6e^{-9} \end{bmatrix}$

b) Find values for $\hat{A}, \hat{B}, \hat{C}$ and x_0 if we use the approximation that $\hat{A} = I + TA$ and T=1 second and calculate the first three values of the zero-input solution for k=1,2, and 3. Compare to your answer to part a)

$$\hat{A} = I + T_s A = \begin{bmatrix} 1-1 & 0 \\ 0 & 1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \hat{B} = T_s B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} and\ \hat{C} = C = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

For k=1,2, and 3:

$$x_{zero-input}(1) = \hat{A}x(0) = \begin{bmatrix} 0 \\ -12 \end{bmatrix}, x_{zero-input}(2) = \hat{A}^2 x(0) = \begin{bmatrix} 0 \\ 24 \end{bmatrix}, x_{zero-input}(3) = \hat{A}^3 x(0) = \begin{bmatrix} 0 \\ -48 \end{bmatrix}$$

These are not very close to the continuous-time values!

c) Repeat part b) using the approximation that, $\hat{A} = e^{AT}$. What conclusions can you make about the two methods of finding a discrete state model from a continuous time state model?

Sol'n:

$$\hat{A} = e^{AT_s} = \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-3} \end{bmatrix}, \hat{C} = C = \begin{bmatrix} 1 & 0 \end{bmatrix}, and$$

$$\hat{B} = \int_0^{T_s} e^{At} dt B = A^{-1}[e^{AT_s} - I]B = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}^{-1} \begin{bmatrix} e^{-1}-1 & 0 \\ 0 & e^{-3}-1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - e^{-1} \\ -3e^{-3} + 3 \end{bmatrix}$$

For k=1,2, and 3:

$$x_{zero-input}(1) = \hat{A}x(0) = \begin{bmatrix} 5e^{-1} \\ 6e^{-3} \end{bmatrix}, x_{zero-input}(2) = \hat{A}^2 x(0) = \begin{bmatrix} 5e^{-2} \\ 6e^{-6} \end{bmatrix}, x_{zero-input}(3) = \hat{A}^3 x(0) = \begin{bmatrix} 5e^{-3} \\ 6e^{-9} \end{bmatrix}$$

These are identical to the continuous time zero-input solution evaluated at 1,2, and 3 seconds!

Conclusion: The 2nd (preferred method) of discretization preserves stability and produces a zero-input invariant model!!!

d) For the SCALAR system, x=ax+bw, with output, y=cx, find the approximate digital transfer function, G(z)=Y(z)/W(z) using:

i) The approximation that $\hat{A} = e^{AT}$

ii) The approximation that $\hat{A} = I + TA$

iii) The bilinear transformation

iv) The correct impulse invariant design method you learned in class today

v) The correct step invariant design method you learned in class today

<u>Solution:</u> *In continuous time, the transfer function is H(s)=c[sI-a]$^{-1}$b = cb/(s-a). If we have a discrete model,* $x_{k+1} = \hat{A}x_k + \hat{B}w_k$ *, is* $H(z) = \hat{C}[zI - \hat{A}]^{-1}\hat{B} = \hat{c}\hat{b}/(z - \hat{a})$.

i) $H(z) = \hat{c}\hat{b}/(z - \hat{a}) = c(a^{-1}(e^{aT} - 1)b)/(z - e^{aT}) = cb(e^{aT} - 1)/a(z - e^{aT})$

ii) $H(z) = \hat{c}\hat{b}/(z - \hat{a}) = Tcb/(z - (1 + Ta))$

iii) $H(z) = H(s)|_{\frac{2(z-1)}{T(z+1)}} = cb/(\frac{2(z-1)}{T(z+1)} - a) = \frac{cb\frac{T}{2}(z+1)}{(1-a\frac{T}{2})z - (1+a\frac{T}{2})} = \frac{cb\frac{T}{2(1-a\frac{T}{2})}(z+1)}{z - \frac{(1+a\frac{T}{2})}{(1-a\frac{T}{2})}}$

iv) H(s)=cb/(s-a). W(s)=1, thus Y(s) = cb/(s-a) and y(t)=cbe^{at}*u(t). y(kT)=cbe*^{akT}*u(kT) and Y(z)=cb/(z-e*^{aT}*) and W(z)=1/T. Thus, H(z)=Y(z)/W(z)= cbT/(z-e*^{aT}*).*

v) H(s)=cb/(s-a). W(s)=1/s, thus Y(s) = cb/(s(s-a)) and y(t)=cb/a(1-e^{at}*)u(t). y(kT)=cb/a(1-e*^{akT}*)u(kT) and Y(z)=cb/a[z/(z-1)-z/(z-e*^{aT}*)] and W(z)=z/z-1. Thus, H(z)=Y(z)/W(z)=cb/a[1-(z-1)/(z-e*^{aT}*)]= cb/a[(1-e*^{aT}*)/(z-e*^{aT}*)]*

e) Are any of your answers for G(z) the same? What if T=1 sec?

The step invariant and the preferred method give the same answer. This is expected since the preferred method assumes that the input is piecewise constant and a step input is definitely piecewise constant!