EE572 Solution to HW#1

1a) The next rabbit population ($P_{k+1}$) will be the old population ($P_k$) plus new born rabbits ($rP_k$) minus any eaten rabbits ($w_k$). Or, putting this all together, we have $P_{k+1}=P_k + rP_k - w_k = (1+r)P_k - w_k$.

1b) In this problem, the state $x_k$ is the population $P_k$. The solution to the standard next state equation $x_{k+1}=ax_k+bw_k$ with initial state $x_0$ is $x_k = a^k x_0 + \sum_{j=1}^{k} a^{k-j} b w_{j-1}$. If the number of eaten rabbits $w_k = w$ is constant, then we can factor it out from the solution to find $P_k = a^k P_0 + (\sum_{j=1}^{k} a^{k-j} b) \cdot w$. In our problem, a=(1+r) and b=-1. To make the population go to zero in 24 months, we must have $P_{24} = 0$. Or, pluging in the values for r=10% which makes a=1.1 we have: $P_{180} = 0 = (1.1)^{24}(100) + \sum_{j=1}^{24} (1.1)^{24-j}(-1)w$. But from the hint, we know that $\sum_{j=1}^{k} a^{k-j} = a^{k-1} \sum_{j=0}^{k-1} (\frac{1}{a})^j = a^{k-1}\left[\frac{(\frac{1}{a})^k - 1}{\frac{1}{a} - 1}\right] = \frac{a^k - 1}{a-1}$ or 88.9473 Solving for w, we find $w = (1.1)^{24}100/88.9473 = 11.13$ rabbits eaten per month. Note that we eat a total of 24x11.13 = 267.12 rabbits to kill off an initial population of 100!

1c) To keep the rabbit population the same, we must eat the new born rabbits. Thus, $w=rP_k = rP_0 = 0.1 \times 100 = 10$ rabbits per month.

1d) The sate is the rabbit population, $P_k$. The input is the number of monthly eaten rabbits, $w_k$.

1e) The system is UNSTABLE (i.e., will blow up if we eat no rabbits)

1f) The system is controllable. We can force the rabbit population to zero (see problem 1b)

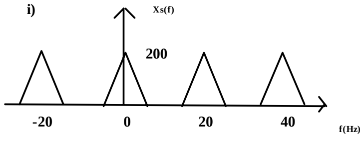
2a) In *ideal* sampling, $x_s(t)=x(t)p(t)$ where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$. But the Fourier Series representation for this periodic impulse train is $p(t) = \sum_{n=-\infty}^{\infty} P_n e^{jn2\pi f_s t}$ where $P_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} p(t) e^{-jn2\pi f_s t} dt = 1/T_s = f_s$. Thus, the Fourier Transform of $x_s(t)$ is

$$X_s(f) = \int_{-\infty}^{\infty} p(t)x(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_s e^{jn2\pi f_s t} x(t) e^{-j2\pi ft} dt = \sum_{n=-\infty}^{\infty} f_s \int_{n=-\infty}^{\infty} x(t) e^{-j2\pi(f-nf_s)t} dt = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

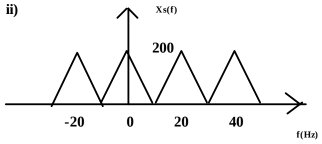
Thus, the Fourier transform of $x_s(t)$ is just the Fourier transform of the original signal repeated every $nf_s$

2b) The sampling frequency is $f_s = 1/(0.05) = 20$ Hz. Thus, the Fourier Transform of $x_s(t)$ is:

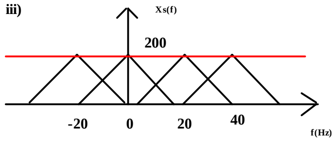
i)



ii)



iii)



2c) To avoid overlap or *aliasing*, as shown in 2b) part iii), the sampling frequency must be at least twice the highest frequency present in the original signal. Or, $f_s > 2$ x highest frequency in x(t). In part i), $f_{nyquist}=2 \text{ x } 5 = 10$ Hz. In part ii), $f_{nyquist}=2 \text{ x } 10 = 20$ Hz. Finally, in part iii) $f_{nyquist}=2 \text{ x } 20 = 40$ Hz. Since in part c), we are not sampling above the nyquist frequency, aliasing (overlap) occurs and the signal is permanently corrupted!

In EE572, we will show that sampling at the Nyquist Frequency is NOT sufficient for closed-loop digital control systems!