EE572 - Solution to HW #18

1.a) Determine the type number of the following open-loop Z-domain transfer functions:

$$i)\,G(z)=\frac{10(z+1)^2}{z(z-1)} \qquad ii)\,G(z)=\frac{10(z+1)^3}{z^2(z-1)} \qquad iii)\,G(z)=\frac{10(z+1)^3}{z(z-1)^2}$$

Solution: i) type 1 ii) type 1 iii) type 2

b) Let T_s=100 msec and find the static error coefficients K_p, K_v and K_a for problem 1a) (assume unity feedback).

Solution: i) $K_p=\lim_{z\to1}G(z)=\infty,\quad K_v=\lim_{z\to1}\frac{z-1}{zTs}G(z)=40/T_s=400,\quad K_a=\lim_{z\to1}\left(\frac{z-1}{zTs}\right)^2G(z)=0$

ii) $K_p=\lim_{z\to1}G(z)=\infty,\quad K_v=\lim_{z\to1}\frac{z-1}{zTs}G(z)=80/T_s=800,\quad K_a=\lim_{z\to1}\left(\frac{z-1}{zTs}\right)^2G(z)=0$

iii) $K_p=\lim_{z\to1}G(z)=\infty,\quad K_v=\lim_{z\to1}\frac{z-1}{zTs}G(z)=\infty,\quad K_a=\lim_{z\to1}\left(\frac{z-1}{zTs}\right)^2G(z)=80/(T_s)^2=8000$

c) For each of the closed-loop unity feedback systems in part a), find:

i)e_{ss} due to a step ii)e_{ss} due to a ramp iii)e_{ss} due to a parabola

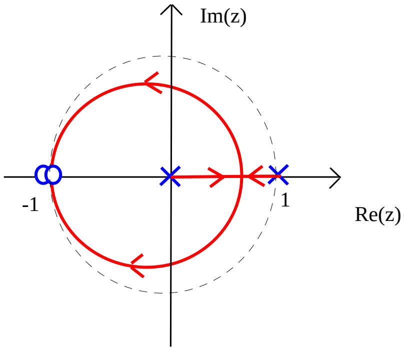
i) e_{ss} due to a step = $1/(1+K_p)$ = 0 (part 1a)i)), = 0 (part 1a)ii)), = 0 (part 1a)iii))

ii) e_{ss} due to a ramp = $1/K_v$ = 1/400 (part 1a)i)), = 1/800 (part 1a)ii)), = 0 (part 1a)iii))

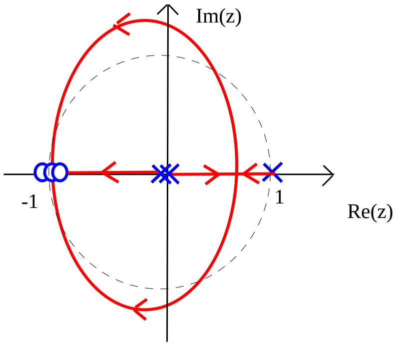
iii) e_{ss} due to a parabola = $1/K_a$ = ∞ (part 1a)i)), = ∞ (part 1a)ii)), = 1/8000 (part 1a)iii))

d) Sketch the root locus for the systems in part a).

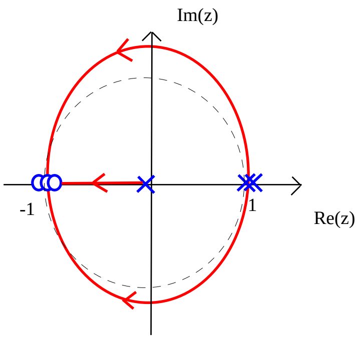
Solution:



Root locus of Prob. 1a) part i)

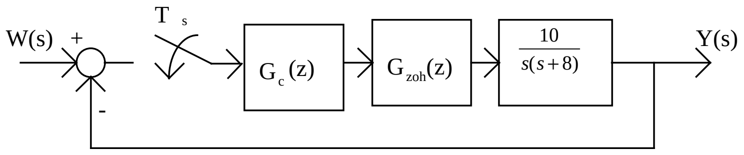


Root locus of Prob. 1a) part ii)



Root locus of Prob. 1a) part iii)

2. Given the system



a) Find a Z-domain model for the open-loop system including the ZOH if T_s = 10 msec.

Solution: $G_{zoh}G(z)=[z^{-1}-1]Z\{G_{zoh}G(s)/s\}=[z^{-1}-1]Z\{10/(s^2(s+8))\}=$ $[z^{-1}-1]1.2019\mathrm{x}10^{-6}(z+1)^3/((z-1)^2(z-0.9231))=1.2019\mathrm{x}10^{-6}(z+1)^3/(z(z-1)(z-0.9231))$

b) What type number is your Z-domain model? What type is the original model?

Solution: since $G_{zoh}G(z)$ has one open-loop pole at z=1, it is a type one system. Since the original $G_{zoh}G(s)$ has one open-loop pole at s=1, it is also a type one system.

c) Find e_{ss} due to a step, e_{ss} due to a ramp, and e_{ss} due to a parabola for your uncompensated model.

Solution:

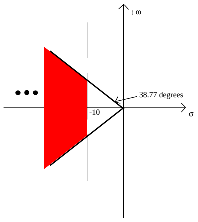
$$K_p = \lim_{z\to 1} G_{zoh}G(z) = \infty, \qquad K_v = \lim_{z\to 1}\frac{z-1}{zTs}G(z) = 1.25\times 10^{-4}/T_s = 1.25\times 10^{-2}, \qquad K_a = \lim_{z\to 1}\left(\frac{z-1}{zTs}\right)^2 G(z) = 0$$

Thus, $e_{ss}|_{step}$ = 1/(1+K_p) = 0, $e_{ss}|_{ramp}$ = 1/K_v = 80, e_{ss} due to a parabola = 1/K_a = ∞

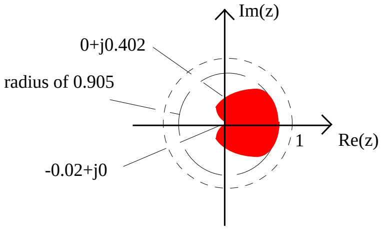
d) Given the following transient specifications: $t_s < 0.4$ sec and $M_p < 2\%$. Illustrate the region of the s-plane and the z-plane where we must place our dominant poles to satisfy these specs.

Solution: from the first spec, we find that $t_s < 0.4 \Rightarrow \zeta\omega_n > 10$. From the second spec we find that $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100\%$. Or, solving this relationship for the damping coefficient we obtain, $\zeta = \sqrt{\frac{\ln(m)^2}{\pi^2+\ln(m)^2}}$ where m=M_p/100% = 0.02. Thus, $\zeta = \sqrt{\frac{\ln(0.02)^2}{\pi^2+\ln(0.02)^2}} = 0.7797 = \cos\theta$. Hence, $\theta = \cos^{-1}(\zeta) = \cos^{-1}(0.7797) = 38.77^\circ$.

Since we must have $M_p < 2\%$, our constraint becomes $\theta \le 38.77^\circ$. In the s-plane, we obtain the following region:

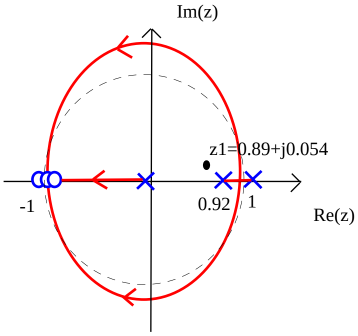


In the z-plane, this region maps to:



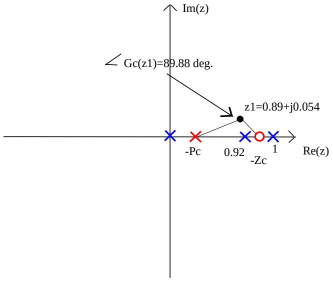
e) Design Gc(z) (plus possibly a lag compensator) to meet the above specs plus the added spec that e_{ss} due to ramp ≤ 1/50.

Solution: We must first design a lead compensator to meet the transient specs. To be consistent, let's pick the desired dominant poles to be $s_1=-11+j6$ which maps to z_1 = 0.8942 + j0.0537. Perhaps the uncompensated root locus will pass thru z1. To check, let's sketch the uncompensated root locus of $G_{zoh}G(z) = 1.2019\times 10^{-6}(z+1)^3/(z(z-1)(z-0.9231))$:



Root locus of GzohG(z)

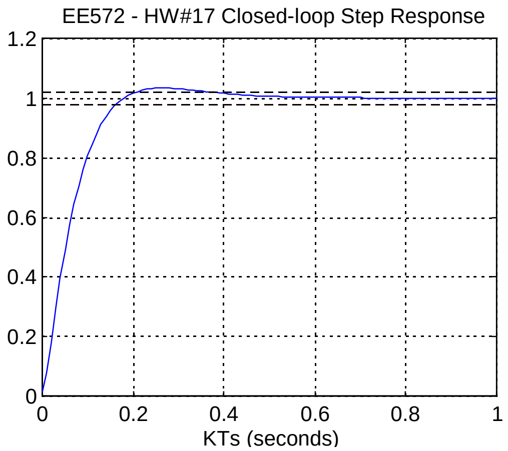
As can be seen, the root locus does not pass thru z_1. Next, let's find the angle of deficiency (i.e., the amount of lead angle needed to bend the root locus thru z_s). The angle of deficiency is $\angle G_{lead}(z_1) = 180° \times odd\# - \angle G_{zoh}G(z_1)|_{z_1=0.89+j0.054}$ = 180^o x odd# - 90.12^o = 89.88^o. Therefore, our lead compensator must supply 89.88 degrees (this compares favorably to the solution in the s-plane which provided 89.77 degrees).

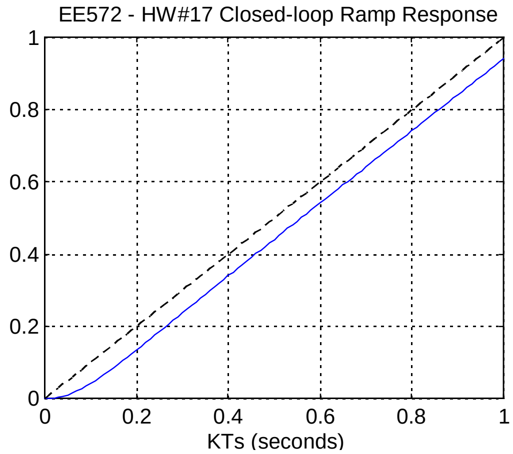


Let's pick $-z_c$ to be as far to the left as possible while still having $-p_c$ to the right of the origin. Let z_c be -0.8976. Then $\angle G_{lead}(z_1) = 89.88^\circ = \angle(z_1 + 0.8976) - \angle(z_1 + p_c) = 92.63^\circ - \angle(z_1 + p_c) \; or \; \angle(z_1 + p_c) = 3.51^\circ$. Solving for p_c: $IM(z_1)/RE(z_1 + p_c) = \tan(2.13^\circ) \, or \; p_c = IM(z_1)/\tan(3.51^\circ) - RE(z_1) = -0.0189$. The last step is to find K_c from the magnitude condition: $K_c = 1/|G_{zoh}G(z_1)(z_1 + z_c)/(z_1 + p_c)| = 12{,}910$. Thus, the lead compensator is $G_{lead}(s)$=12,910(z-0.8976)/(z-0.0189). Finally, after performing the lead compensator design, we can find a lag compensator to put in series with G_{lead}. The desired K_v is 50 to meet the e_{ss} specs. Thus, $K_v = \lim_{z \to 1} \frac{z-1}{T_s z} G_c G_{zoh} G(z) = 0.1684/T_s = 16.84$. Thus, we need to increase the steady-state gain by a factor of 50/16.84=2.9694 to meet our e_{ss} specs. Therefore, Let $(1+z_{lag})/(1+p_{lag})$ = 2.986. Let's pick p_{lag} = -0.9999. Thus, z_{lag}=-0.9997. To find K_{lag}, use the magnitude condition: $K_{lag} = 1/|(\frac{z-0.9997}{z-0.9999}) G_c G_{zoh} G(z)|_{z=z_1=0.89+j0.054} = 1.0015$. Therefore, the lag compensator which must be inserted in series with $G_{lead}(z)$ is $G_{lag}(z)$ = 1.0015(z-0.9997)/(z-0.9999).

f) Simulate a step and ramp response of your closed-loop compensated digital system using dlsim() in MATLAB. Measure t_s, M_p, and e_{ss} (both step and ramp).

Solution: The closed-loop step and ramp responses are shown below:





g) How do your digital compensator numbers compare to the Gc(z) we designed for this system in HWs #15 and #16?

Solution: The numbers are almost identical except for K_{lead} which is about 100 times greater.