

1a) The solution is $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bw(\tau)d\tau$. To evaluate e^{At} , first let's find the eigenvalues of A:

$$A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \Rightarrow |sI - A| = \begin{vmatrix} s+4 & -2 \\ -2 & s+4 \end{vmatrix} = s^2 + 8s + 12 = (s+2)(s+6) = 0 \quad \therefore s_1 = -2 \text{ and } s_2 = -6$$

$$\text{Next, by the Cayley-Hamilton Theorem, } e^{At} = c_0A^0 + c_1A^1 = c_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_1 \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} c_0 - 4c_1 & 2c_1 \\ 2c_1 & c_0 - 4c_1 \end{bmatrix}$$

To find the unknown coefficients, we must evaluate $e^{s_i t} = c_0s_i^0 + c_1s_i^1$ for

$$i=1,2. \begin{bmatrix} e^{-2t} \\ e^{-4t} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} e^{-2t} \\ e^{-4t} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ e^{-4t} \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ 1/2e^{-2t} - 1/2e^{-4t} \end{bmatrix}$$

$$\text{Therefore, } e^{At} = \begin{bmatrix} c_0 - 4c_1 & 2c_1 \\ 2c_1 & c_0 - 4c_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-2t} + e^{-6t} & e^{-2t} - e^{-6t} \\ e^{-2t} - e^{-6t} & e^{-2t} + e^{-6t} \end{bmatrix}$$

Hence, the zero-input portion of our solution is $x(t) = e^{A(t+3)}x(3) =$

$$z_{\text{zero-input}}(t) = \frac{1}{2} \begin{bmatrix} e^{-2(t+3)} + e^{-6(t+3)} & e^{-2(t+3)} - e^{-6(t+3)} \\ e^{-2(t+3)} - e^{-6(t+3)} & e^{-2(t+3)} + e^{-6(t+3)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-2(t+3)} + e^{-6(t+3)} \\ e^{-2(t+3)} - e^{-6(t+3)} \end{bmatrix}$$

$$\text{The zero-state solution is } x_{\text{zero-state}}(t) = \int_{t_0}^t e^{A(t-\tau)}Bw(\tau)d\tau = \frac{1}{2} \int_{-3}^t \begin{bmatrix} e^{-2(t-\tau)} + e^{-6(t-\tau)} & e^{-2(t-\tau)} - e^{-6(t-\tau)} \\ e^{-2(t-\tau)} - e^{-6(t-\tau)} & e^{-2(t-\tau)} + e^{-6(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\tau}u(\tau)d\tau$$

$$= \frac{1}{2} \int_0^t \begin{bmatrix} e^{-2(t-\tau)} - e^{-6(t-\tau)} \\ e^{-2(t-\tau)} + e^{-6(t-\tau)} \end{bmatrix} e^{-\tau}d\tau u(t) = \frac{1}{2} \int_0^t \begin{bmatrix} e^{-2t+\tau} - e^{-6t+5\tau} \\ e^{-2t+\tau} + e^{-6t+5\tau} \end{bmatrix} d\tau u(t) = \frac{1}{2} \begin{bmatrix} e^{-2t+\tau} - \frac{1}{5}e^{-6t+5\tau} \\ e^{-2t+\tau} + \frac{1}{5}e^{-6t+5\tau} \end{bmatrix}^t_0 u(t) = \frac{1}{2} \begin{bmatrix} e^{-t} - \frac{1}{5}e^{-t} - e^{-2t} + \frac{1}{5}e^{-6t} \\ e^{-t} + \frac{1}{5}e^{-t} - e^{-2t} - \frac{1}{5}e^{-6t} \end{bmatrix} u(t)$$

$$\text{Therefore, the solution is } x(t) = \frac{1}{2} \begin{bmatrix} e^{-2(t+3)} + e^{-6(t+3)} \\ e^{-2(t+3)} - e^{-6(t+3)} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 4/5e^{-t} - e^{-2t} + 1/5e^{-6t} \\ 6/5e^{-t} - e^{-2t} - 1/5e^{-6t} \end{bmatrix} u(t)$$

$$1b) \text{ The output is } y(t) = [1 \ 0]x + [2]w(t) = \frac{1}{2}(e^{-2(t+3)} + e^{-6(t+3)}) + [4/5e^{-t} - e^{-2t} + 1/5e^{-6t}]u(t) + 2e^{-t}u(t)$$

1c) The transfer function is $Y(s)/W(s) = C[sI - A]^{-1}B + D =$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & -2 \\ -2 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [2] = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(s+2)(s+6)} \begin{bmatrix} s+4 & 2 \\ 2 & s+4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [2] = \frac{2}{(s+2)(s+6)} + 2 = \frac{2s^2 + 16s + 14}{(s+2)(s+6)}$$

1d) i) eigenvalues are $\{0, -2\}$, therefore marginally stable

ii) eigenvalues are $\{0, 0\}$, therefore unstable

iii) eigenvalues are $\{-1, -1\}$, therefore asymptotically stable

iv) eigenvalues are $\{1, -1\}$, therefore unstable

1e) i) poles are $\{-1, -2\}$, therefore asymptotically stable

ii) poles are $\{-1, 2\}$, therefore unstable

iii) poles are $\{0, -1\}$, therefore marginally stable

iv) poles are $\{0, 0, -1\}$, therefore unstable

1f) Using the techniques we learned in class, we obtain the following block diagram:

