1a) The solution to the state equation is $x(t)=e^{At}\,x(0)$. To evaluate, first let's find the eigenvalues of A:

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\Rightarrow |sI-A|=\begin{vmatrix}s-2 & -1\\ -1 & s-2\end{vmatrix}=s^2-4s+3=(s-1)(s-3)=0\ \therefore s_1=1\ \ and\ \ s_2=-3$$

Next, let's find the eigenvectors of A:

$$[s_1I-A]\underline{P}_1=\begin{bmatrix}1-2 & -1\\ -1 & 1-2\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}p_{11}\\ p_{12}\end{bmatrix}=0\ \therefore \underline{P}_1=\begin{bmatrix}p_{11}\\ p_{12}\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$[s_2I-A]\underline{P}_2=\begin{bmatrix}3-2 & -1\\ -1 & 3-2\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}p_{21}\\ p_{22}\end{bmatrix}=0\ \therefore \underline{P}_2=\begin{bmatrix}p_{21}\\ p_{22}\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Therefore $P=[\underline{P}_1\quad \underline{P}_2]=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ and $S=\begin{bmatrix}s_1 & 0\\ 0 & s_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$. Thus, we can $e^{At}=P\,e^{St}\,P^{-1}=$

$$\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{s_1t} & 0\\ 0 & e^{s_1t}\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^{t} & 0\\ 0 & e^{3t}\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^t+e^{3t} & -e^t+e^{3t}\\ -e^t+e^{3t} & e^t+e^{3t}\end{bmatrix}$$

Hence, the solution is $x(t)=e^{At}\,x(0)=$

$$\frac{1}{2}\begin{bmatrix}e^t+e^{3t} & -e^t+e^{3t}\\ -e^t+e^{3t} & e^t+e^{3t}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^t+e^{3t}\\ -e^t+e^{3t}\end{bmatrix}\ \text{and the output is}\ y=[1\quad 0]x(t)=0.5(e^t+e^{3t})$$

1b) Again, the solution is $x(t)=e^{At}\,x(0)$. To evaluate, first let's find the eigenvalues of A:

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\Rightarrow |sI-A|=\begin{vmatrix}s-2 & -1\\ -1 & s-2\end{vmatrix}=s^2-4s+3=(s-1)(s-3)=0\ \therefore s_1=1\ \ and\ \ s_2=-3$$

Next, by the Cayley-Hamilton Theorem, $e^{At}=c_0A^0+c_1A^1=c_0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+c_1\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}c_0+2c_1 & c_1\\ c_1 & c_0+2c_1\end{bmatrix}$

To find the unknown coefficients, we must evaluate $e^{s_it}=c_0s_i^{\,0}+c_1s_i^{\,1}$ for

i=1,2. $\begin{bmatrix}e^t\\ e^{3t}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}c_0\\ c_1\end{bmatrix}\Rightarrow\begin{bmatrix}c_0\\ c_1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}e^t\\ e^{3t}\end{bmatrix}=\begin{bmatrix}3/2 & -1/2\\ -1/2 & 1/2\end{bmatrix}\begin{bmatrix}e^t\\ e^{3t}\end{bmatrix}=\begin{bmatrix}3e^t-e^{3t}\\ -e^t+e^{3t}\end{bmatrix}\frac{1}{2}$

Therefore, $e^{At}=\begin{bmatrix}c_0+2c_1 & c_1\\ c_1 & c_0+2c_1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^t+e^{3t} & -e^t+e^{3t}\\ -e^t+e^{3t} & e^t+e^{3t}\end{bmatrix}$

Hence, the solution is $x(t)=e^{At}\,x(0)=$

$$\frac{1}{2}\begin{bmatrix}e^t+e^{3t} & -e^t+e^{3t}\\ -e^t+e^{3t} & e^t+e^{3t}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^t+e^{3t}\\ -e^t+e^{3t}\end{bmatrix}\ \text{and the output is}\ y=[1\quad 0]x(t)=0.5(e^t+e^{3t})$$

1c) The solution is $x(t)=e^{At}\,x(0)$. To evaluate, note that the eigenvalues of A are:

$$A=\begin{bmatrix}-4 & 0\\ 3 & -4\end{bmatrix}\Rightarrow |sI-A|=\begin{vmatrix}s+4 & 0\\ -3 & s+4\end{vmatrix}=(s+4)(s+4)=0\quad \therefore s_1=-4\ and\ s_2=-4$$

Next, by the Cayley-Hamilton Theorem, $e^{At} = c_0A^0 + c_1A^1 = c_0\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + c_1\begin{bmatrix}-4 & 0\\3 & -4\end{bmatrix} = \begin{bmatrix}c_0-4c_1 & 0\\3c_1 & c_0-4c_1\end{bmatrix}$

To find the unknown coefficients, ordinarily we evaluate $e^{s_it} = c_0s_i^{\,0} + c_1s_i^{\,1}$ for i=1,2

However, in this case we have repeated eigenvalues and we would only obtain the same equation twice! So, to obtain a linearly independent equation, let us take the derivative with respect to s_i:

$\frac{d}{ds_i}(e^{s_it}) = te^{s_it} = \frac{d}{ds_i}(c_0s_i^{\,0} + c_1s_i^{\,1}) = 0 + c_1$. Therefore,

$$\begin{bmatrix}e^{-4t}\\te^{-4t}\end{bmatrix} = \begin{bmatrix}1 & -4\\0 & 1\end{bmatrix}\begin{bmatrix}c_0\\c_1\end{bmatrix} \Rightarrow \begin{bmatrix}c_0\\c_1\end{bmatrix} = \begin{bmatrix}1 & -4\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}e^{-4t}\\te^{-4t}\end{bmatrix} = \begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}\begin{bmatrix}e^{-4t}\\te^{-4t}\end{bmatrix} = \begin{bmatrix}1e^{-4t}+4te^{-4t}\\te^{-4t}\end{bmatrix}$$

Substituting into our State Transition matrix we find, $e^{At} = \begin{bmatrix}c_0-4c_1 & 0\\3c_1 & c_0-4c_1\end{bmatrix} = \begin{bmatrix}e^{-4t} & 0\\3te^{-4t} & e^{-4t}\end{bmatrix}$

Hence, the solution is $x(t)=e^{At}\,x(0)= \begin{bmatrix}e^{-4t} & 0\\3te^{-4t} & e^{-4t}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}e^{-4t}\\3te^{-4t}+2e^{-4t}\end{bmatrix}$

1d) First, let the characteristic equation of the matrix A to be:
$|sI-A| = c_0s^0 + c_1s^1 + ... + c_{n-1}s^{n-1} + s^n = 0$.
To prove that $A^k = a_0A^0 + a_1A^1 + ... + a_{n-1}A^{n-1}$, for all k>0, let us first show that it is true for k=1:
$A^1 = a_0A^0 + a_1A^1 + ... + a_{n-1}A^{n-1} = 0A^0 + 1A^1 + 0A^2 + 0A^3 + ... + 0A^{n-1}$
or $a_1=1$ and $0= a_0 = a_2 = a_3 = a_4 = \ldots = a_{n-1}$. Next, let us assume that our relationship is true for k=m. That is, there exists coefficients b_i i=1,2,…,n such that $A^m = b_0A^0 + b_1A^1 + ... + b_{n-1}A^{n-1}$. Now, we want to show this true for k=m+1. Let us multiply both sides by A:
$A(A^m) = A^{m+1} = A(b_0A^0 + b_1A^1 + ... + b_{n-1}A^{n-1}) = b_0A^1 + b_1A^2 + ... + b_{n-2}A^{n-1} + b_{n-1}A^n$
But, by the Cayley-Hamilton theorem, A satisfies its own characteristic equation. That is,
$c_0A^0 + c_1A^1 + ... + c_{n-1}A^{n-1} + A^n = 0$ or solving for A^n: $A^n = -c_0A^0 - c_1A^1 - ... - c_{n-1}A^{n-1}$.
Let us substitute this result for A^n into our expression for A^{m+1}:
$A^{m+1} = b_0A^1 + b_1A^2 + ... + b_{n-2}A^{n-1} + b_{n-1}A^n = b_0A^1 + b_1A^2 + ... + b_{n-2}A^{n-1} + b_{n-1}(-c_0A^0 - c_1A^1 - ... - c_{n-1}A^{n-1})$
Note that we now have an expression for A^{m+1} which involves the powers of A from 0 to n-1 and, by virtue of Math Induction, we have proven our result.

2a) i) $L\{\dot f\} = \int_0^\infty \dot f e^{-st}dt = -sf(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty fe^{-st}dt = e^{-s\infty}f(\infty) - f(0^+) + sF(s)$

If we assume that Re(s)>0, then $e^{-\infty s}=0$ and we're done!

ii) Recall that $L\{\dot f\} = \int_0^\infty \dot f e^{-st}dt = sF(s) - f(0)$

Therefore, $\lim_{s\to 0} sF(s) - f(0) = \lim_{s\to 0}\int_0^\infty \dot f e^{-st}dt = \int_0^\infty \dot f e^{-0t}dt = f(\infty) - f(0) \Rightarrow \lim_{s\to 0} sF(s) = f(\infty) = \lim_{t\to\infty} f(t)$

2b) Taking the Laplace transform of both sides, we obtain: $sX(s)-x(0)=AX(s)$
or $X(s)=[sI-A]^{-1}x(0)=$

$$\begin{bmatrix} s+4 & 2 \\ 2 & s+4 \end{bmatrix}^{-1} x(0) = \frac{1}{s^2+8s+12}\begin{bmatrix} s+4 & -2 \\ -2 & s+4 \end{bmatrix} x(0) = \begin{bmatrix} \frac{s+4}{(s+2)(s+6)} & \frac{-2}{(s+2)(s+6)} \\ \frac{-2}{(s+2)(s+6)} & \frac{s+4}{(s+2)(s+6)} \end{bmatrix} x(0)$$

$$= \begin{bmatrix} \frac{1/2}{(s+2)} + \frac{1/2}{(s+6)} & \frac{-1/2}{(s+2)} + \frac{1/2}{(s+6)} \\ \frac{-1/2}{(s+2)} + \frac{1/2}{(s+6)} & \frac{1/2}{(s+2)} + \frac{1/2}{(s+6)} \end{bmatrix} x(0)$$

Now, we may take the inverse Laplace Transform of the above expression to find:

$$x(t) = L^{-1}\{X(s)\} = \frac{1}{2}\begin{bmatrix} e^{-2t} + e^{-6t} & -e^{-2t} + e^{-6t} \\ -e^{-2t} + e^{-6t} & e^{-2t} + e^{-6t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-2t} + e^{-6t} \\ -e^{-2t} + e^{-6t} \end{bmatrix}$$

and the output is $y(t)=[1\ 0]x(t)=1/2(e^{-2t}+e^{-6t})$

2c) Again, taking the Laplace transform of both sides, we obtain: sX(s)-x(0)=AX(s) or $X(s)=[sI-A]^{-1}x(0)=$

$$\begin{bmatrix} s+4 & 0 \\ -3 & s+4 \end{bmatrix}^{-1} x(0) = \frac{1}{(s+4)^2}\begin{bmatrix} s+4 & 0 \\ 3 & s+4 \end{bmatrix} x(0) = \begin{bmatrix} \frac{1}{(s+4)} & 0 \\ \frac{3}{(s+4)^2} & \frac{1}{(s+4)} \end{bmatrix} x(0)$$

Now, we may take the inverse Laplace Transform of the above expression to find:

$$x(t) = L^{-1}\{X(s)\} = \begin{bmatrix} e^{-4t} & 0 \\ 3te^{-4t} & e^{-4t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ 3te^{-4t} + 2e^{-4t} \end{bmatrix}$$

2d) Take Laplace Transforms to solve the following state variable model:

$$sX(s) - x(0) = -2X(s) + 4W(s) \text{ or } X(s) = \frac{x(0)}{s+2} + \frac{4W(s)}{s+2}$$

$$X(s) = \frac{10}{s+2} + \frac{4}{s(s+2)} = \frac{10}{s+2} + \frac{2}{s} - \frac{2}{s+2}$$

Taking the inverse Laplace Transform produces:

$$x(t) = x_{zero-input} + x_{zero-state} = 10e^{-2t} + 2(1 - e^{-2t})u(t)$$

2e) Rather than re-work problem 2d) using Laplace Transforms, we can use the properties of linearity and time invariance on each portion of the solution, $x_{zero-input}+x_{zero-state}$. Notice that the initial state is now half of what it was and has been shifted (delayed) until t_0=2 seconds. This implies that the new zero-input solution will be scaled by 0.5 and delayed by 2 seconds. Also, note that the new input is just the sum of 3 times the old input plus 5 times the old input delayed by 4 seconds. Thus, the new zero-state solution will be simply 3 times the old zero-state solution plus 5 times the old zero-state solution delayed by 4 seconds. That is:

$$x(t) = 0.5x_{zero-input}(t-2) + 3x_{zero-state}(t) + 5x_{zero-state}(t-4) = 5e^{-2(t-2)} + 6(1-e^{-2t})u(t) + 10(1-e^{-2(t-4)})u(t-4)$$