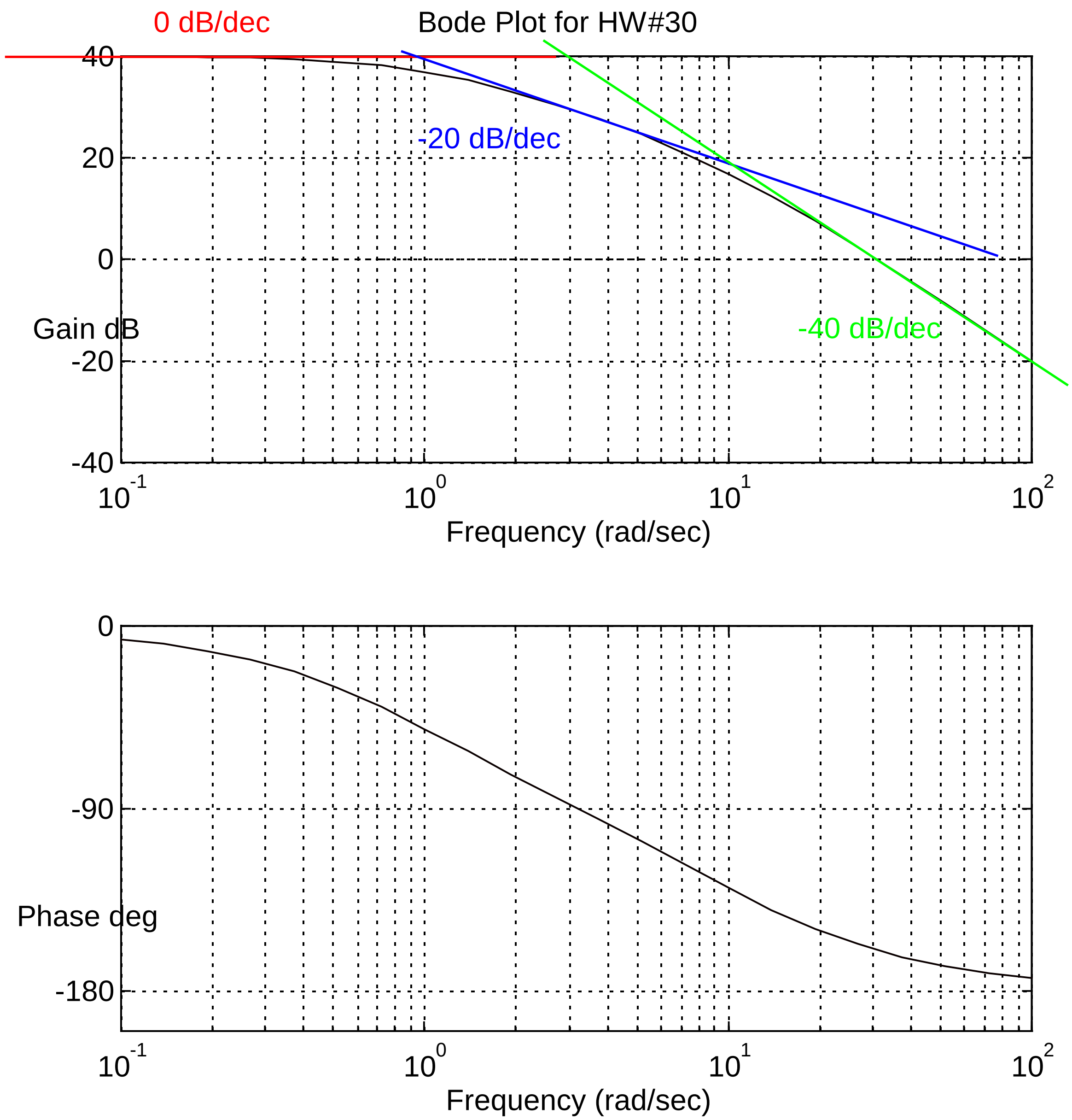
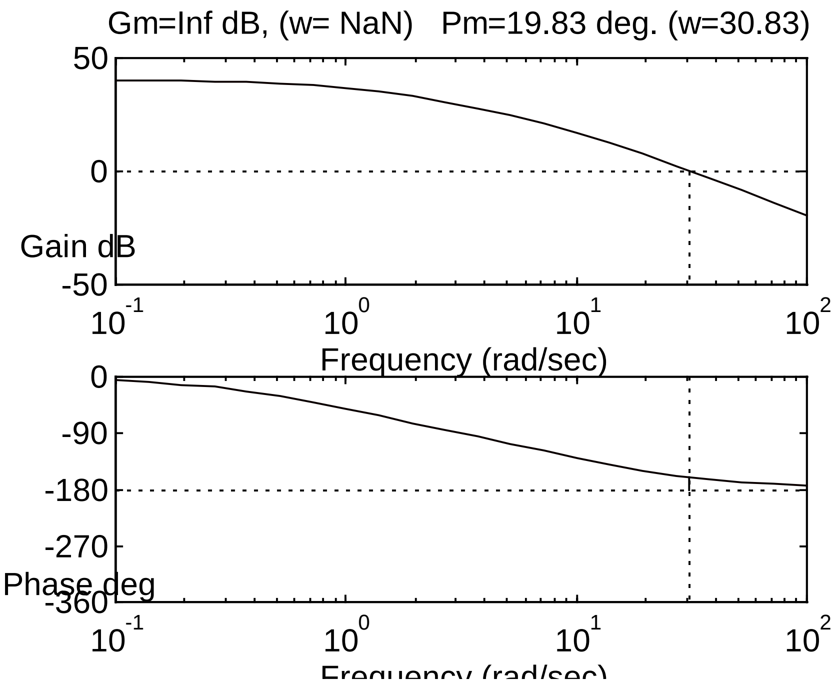
1. a) From the Bode Plot below, find G(s), the open-loop transfer function (assume H(s)=1)



Solution: We see immediately that the system is a type 0 system. From the asymptotes, we find that G(s) has the form G(s) = K/((s/1+1)(s/10+1)). Finally, K= 40 dB = 100. Thus, G(s) = 100/((s/1+1)(s/10+1)).

b) For the system in part b), find $ess|_{step}$ and $ess|_{ramp}$, the gain cross-over frequency (ω_{cg}), the phase cross-over frequency (ω_{cp}), the gain margin (gm) and the phase marging (pm).

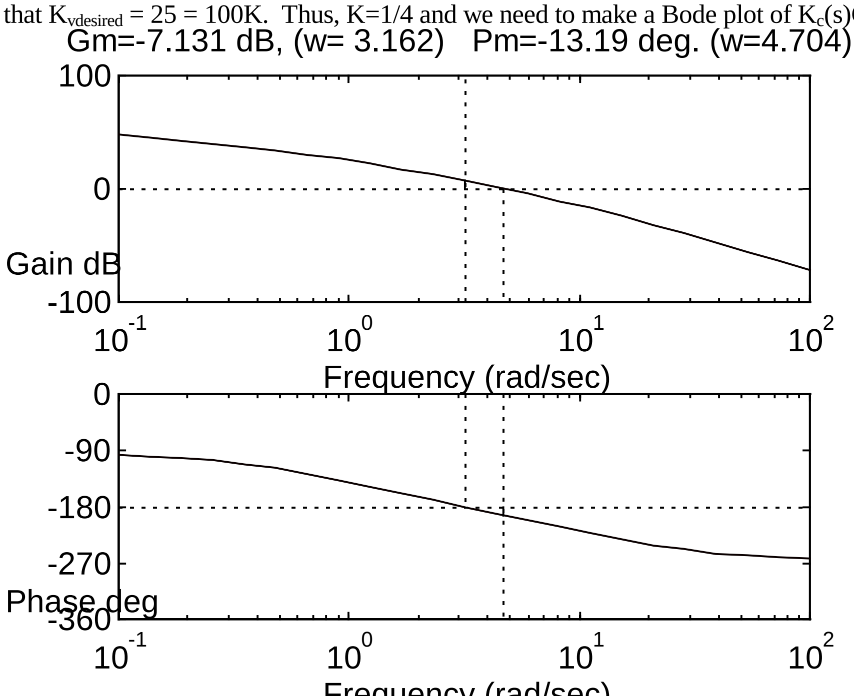
Solution: $ess|_{step} = 1/(1+K_p) = 1/101$, $ess|_{ramp} = 1/K_v = \infty$. The gain cross-over frequency is ω_{cg} = 31.6 rad/sec and (ω_{cg}), the phase cross-over frequency is $\omega_{cp} = \infty$. The gain margin is gm = ∞ and the phase margin is pm = 19.83 degrees (see Matlab plot below):



c) Design a lead compensator of the form, $G_c(s) = K_c(s)(Ts+1)/(\alpha Ts+1)$, such that the compensated system satisfies the following specifications:

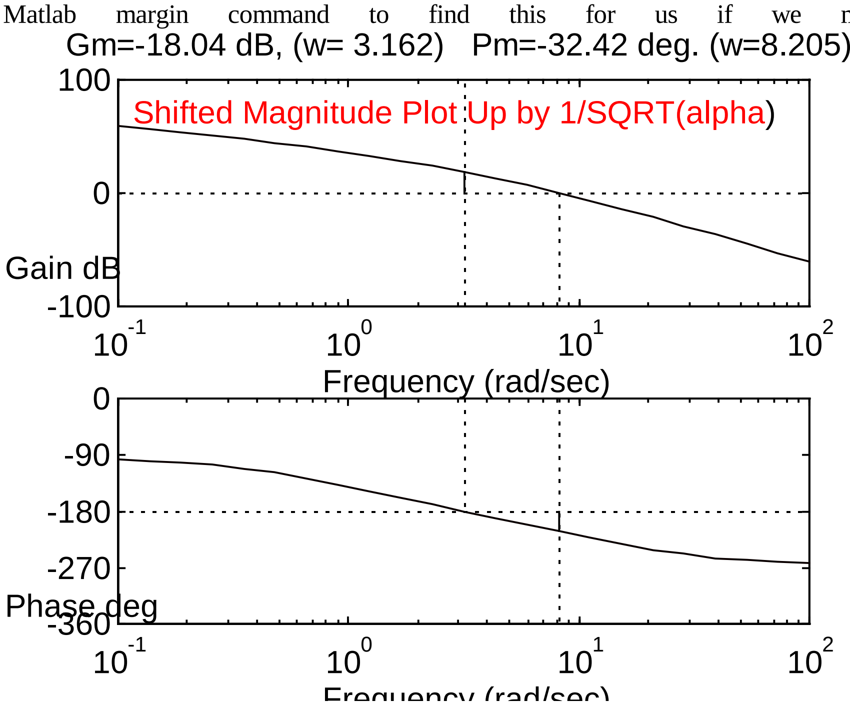
i) gm > 5 dB
ii) pm . 30^o
iii) $ess|_{ramp} = 1/25$

<u>Solution:</u> If the lead compensator is of the form, $G_c(s) = K_c(s)(Ts+1)/(\alpha Ts+1)$, then we must first solve for $K_c(s)$ to meet steady-state error specs. If we want $ess|_{ramp} = 1/25$, we must have a type 1 system. Therefore, $K_c(s)=K/s$ and $ess|_{ramp} = 1/K_{vdesired} = 1/25$ which implies that $K_{vdesired} = 25 = 100K$. Thus, K=1/4 and we need to make a Bode plot of $K_c(s)G = 25//(s(s/1+1)(s/10+1))$.

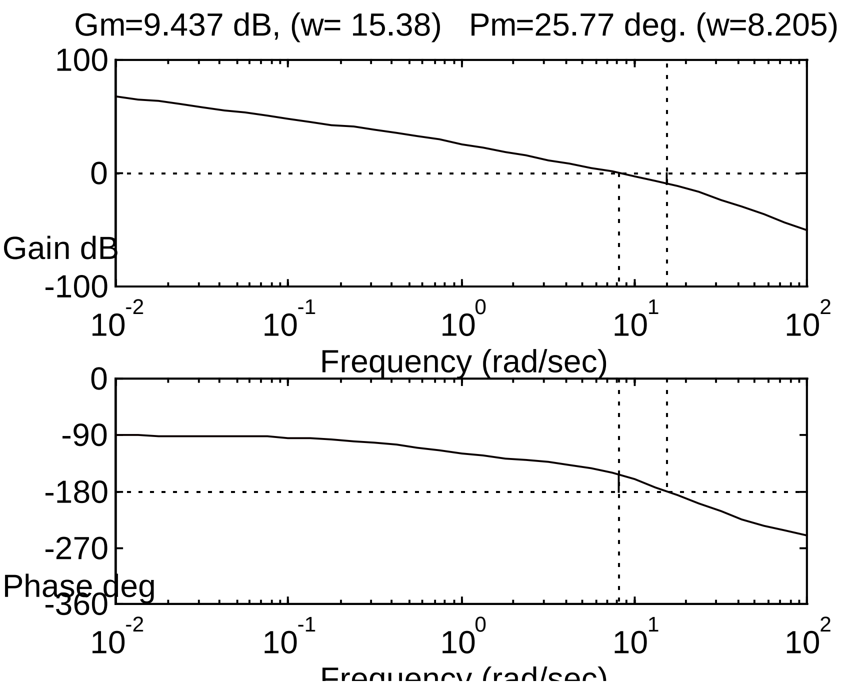


The gain cross-over frequency is ω_{cg} = 4.704 rad/sec and the phase cross-over frequency is ω_{cp} = 3.162 rad/sec. The correspondin gain margin is gm = -7.131 dB and the phase margin is pm = -13.19 degrees (see Matlab plot above). Thus, we need to add in phase of ϕ_m = $p.m._{desired} - p.m._{current}$ + fudge = 30 - (-13.19) + 15 degrees = 58.19 degrees (Note: This is a rather large amount of phase to be added in. Usually, we can comfortably get 45 degrees or so out of a practical lead compensator!)

Once ϕ_m is known, we can find α from $\alpha = \dfrac{1-\sin(\phi_m)}{1+\sin(\phi_m)}$ = 0.0812. Next, in addition to adding in phase, our compensator will add in a gain of $1/\sqrt{\alpha}$ = 3.5049 = 10.89 dB. So, our gain cross-over frequency will shift to the point where $K_c(s)G$ = -10.89 dB. We can use the Matlab margin command to find this for us if we multiply $K_c(s)G$ by $1/\sqrt{\alpha}$ = 3.5049 first:



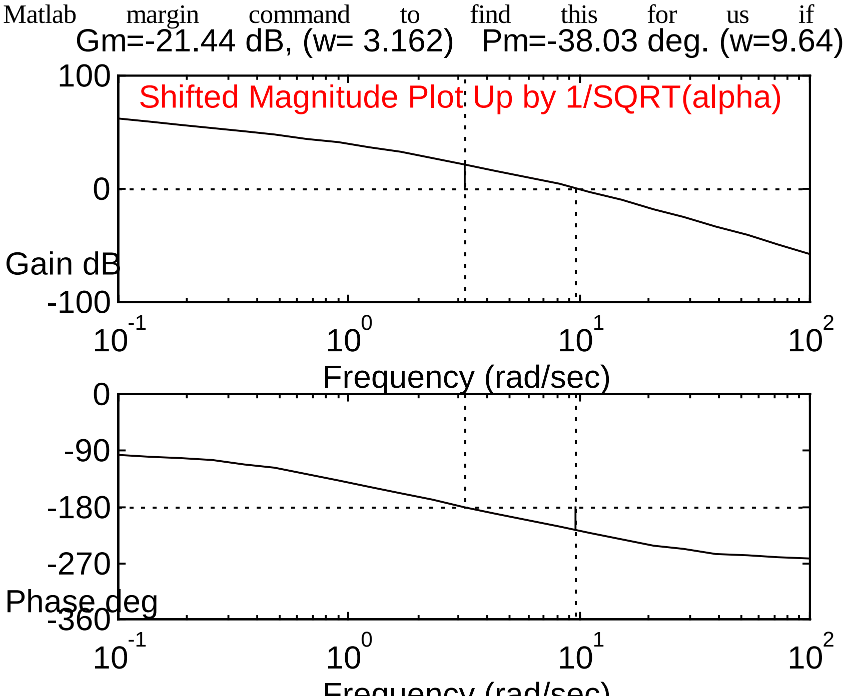
Note that the new gain cross-over frequency will occur at ω_{cgnew} = 8.205 rad/sec. To find the last unknown parameter in our design, we set $\omega_m = \omega_{cgnew} = 1/(\sqrt{\alpha}T)$. Thus, T=0.4227 and our complete compensator looks like $G_c(s) = K_c(s)(Ts+1)/(\alpha Ts+1)$ = 0.25/s[(0.4227s+1)/(0.0347s+1)]. As a final check, consider the Bode Plot of $G_cG(s)$ = 25(0.4227s+1)/[s(s+1)(0.1s+1)(0.0347s+1)]:



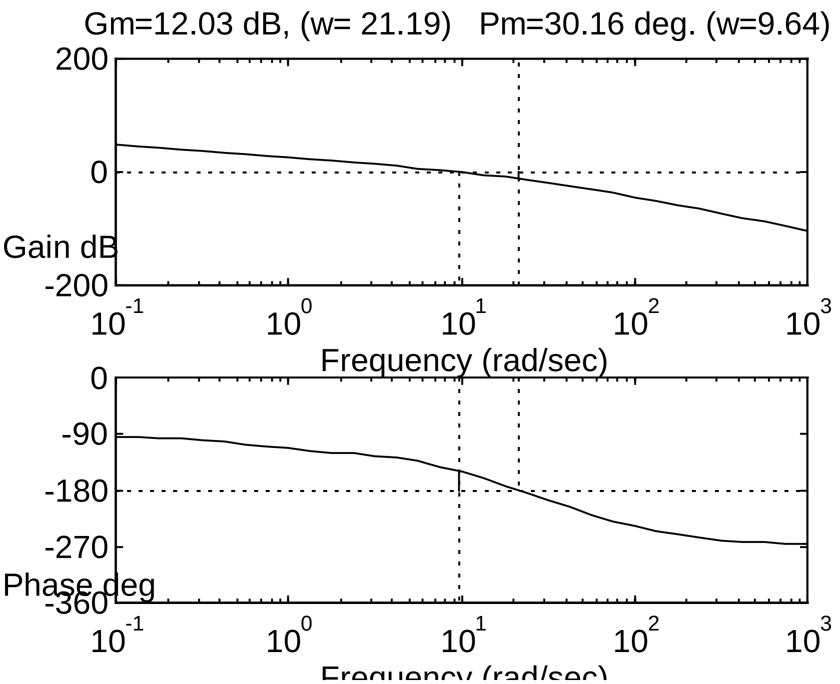
Note that while the new gain cross-over frequency is ω_{cgnew} = 8.205 rad/sec, our phase margin is just slightly below specs at 25.77 degrees. This is because of the large ϕ_m which we needed that shifted the gain cross-over frequency so far to the right (remember, the phase of our system is steadily dropping the higher (more to the right) we look). Thus, let's take one more iteration and try to increase our fudge factor to 25 degrees instead of 15 degrees. Thus, we will need to add in phase of ϕ_m = p.m.$_{desired}$ - p.m.$_{current}$ + fudge = 30

- (-13.19) + 25 degrees = 68.19 degrees (this really is a large amount of phase for a first order compensator to apply. I would probably choose a second order compensator instead)

Once ϕ_m is known, we can find α from $\alpha = \dfrac{1-\sin(\phi_m)}{1+\sin(\phi_m)}$ = 0.0371. Next, in addition to adding in phase, our compensator will add in a gain of $1/\sqrt{\alpha}$ = 5.1905= 14.30 dB. So, our gain cross-over frequency will shift to the point where $K_c(s)G$ = -14.30 dB. We can use the Matlab margin command to find this for us if we multiply $K_c(s)G$ by 5.1905 first:



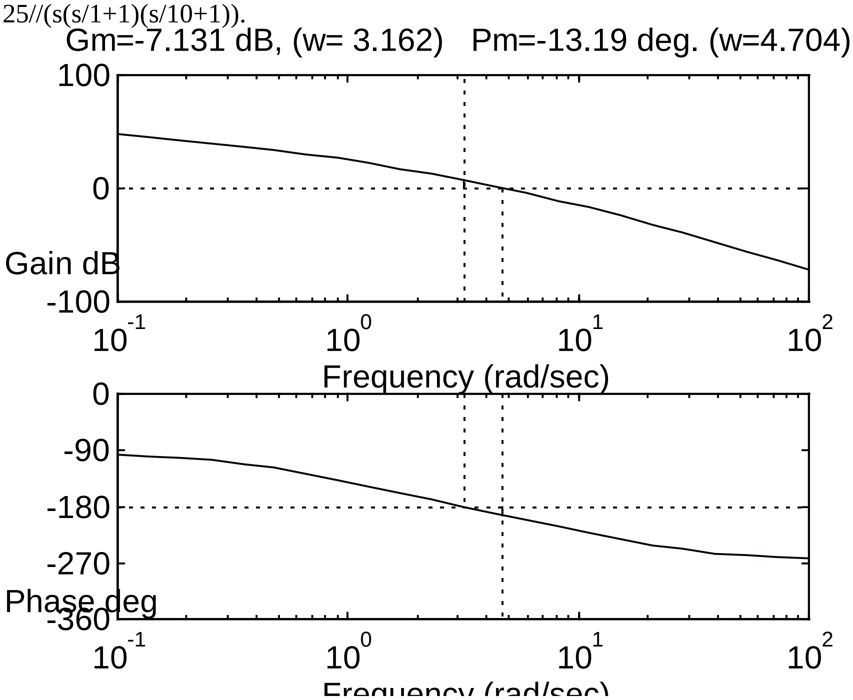
Note that the new gain cross-over frequency will occur at ω_{cgnew} = 9.64 rad/sec. To find the last unknown parameter in our design, we set $\omega_m = \omega_{cgnew} = 1/(\sqrt{\alpha}T)$. Thus, T=0.5384 and our complete compensator looks like $G_c(s)$ = $K_c(s)(Ts+1)/(\alpha Ts+1)$ = 0.25/s[(0.5384s+1)/(0.0200s+1)]. As a final check, consider the Bode Plot of $G_cG(s)$ = 25(0.5384s+1)/[s(s+1)(0.1s+1) (0.0200s+1)].



Our compensated system now meets specs. The gm=12.03 dB at ω_{cp} =21.19 rad/sec and the p.m. is 30.16 degrees at ω_{cg} = 9.64 rad/sec.

d) Repeat using a Lag compensator

The first step of the lag compensator design is identical to the lead design. Note that $G_{Lag}(s) = K_c(s)(Ts+1)/(\beta Ts+1)$, then we must first solve for $K_c(s)$ to meet steady-state error specs. If we want $ess|_{ramp} = 1/25$, we must have a type 1 system. Therefore, $K_c(s)=K/s$ and $ess|_{ramp} = 1/K_{vdesired} = 1/25$ which implies that $K_{vdesired} = 25 = 100K$. Thus, K=1/4 and we need to make a Bode plot of $K_c(s)G = 25//(s(s/1+1)(s/10+1))$.



Now, to perform lag compensation we need to find a new gain cross-over frequency such that the system itself has sufficient phase to meet the phase margin specs. Or, we must find where $\angle K_c G(j\omega_{cgneww}) = -180° + pm_{desired} + fudge$ = $-180° + 30° + 5° = -145°$. The easiest way to find this frequency is to "fool" the margin command into finding this point by subtracting 35 degrees from the phase response:

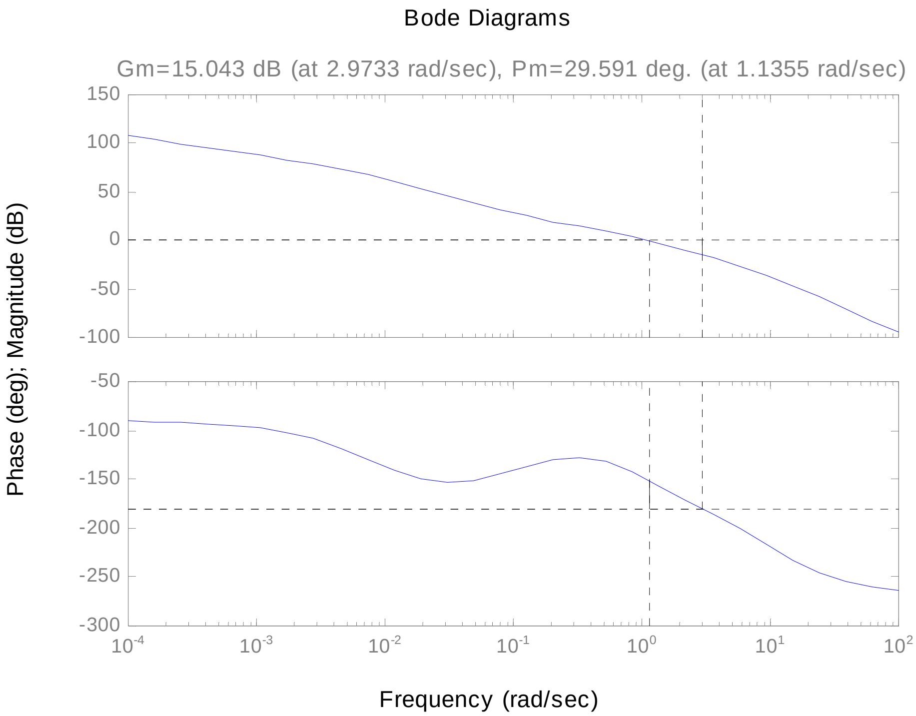
```
» num=25

num =

    25

» den=conv([1 1],[.1 1 0])

den =

    0.1000    1.1000    1.0000         0

» w=logspace(-1,2,200);
» [mag,phs]=bode(num,den,w);
» [gm,pm,wcp,wcg]=margin(mag,phs-35,w)

gm =

    0.0688


pm =

  -48.1907


wcp =
```

1.1320

wcg =

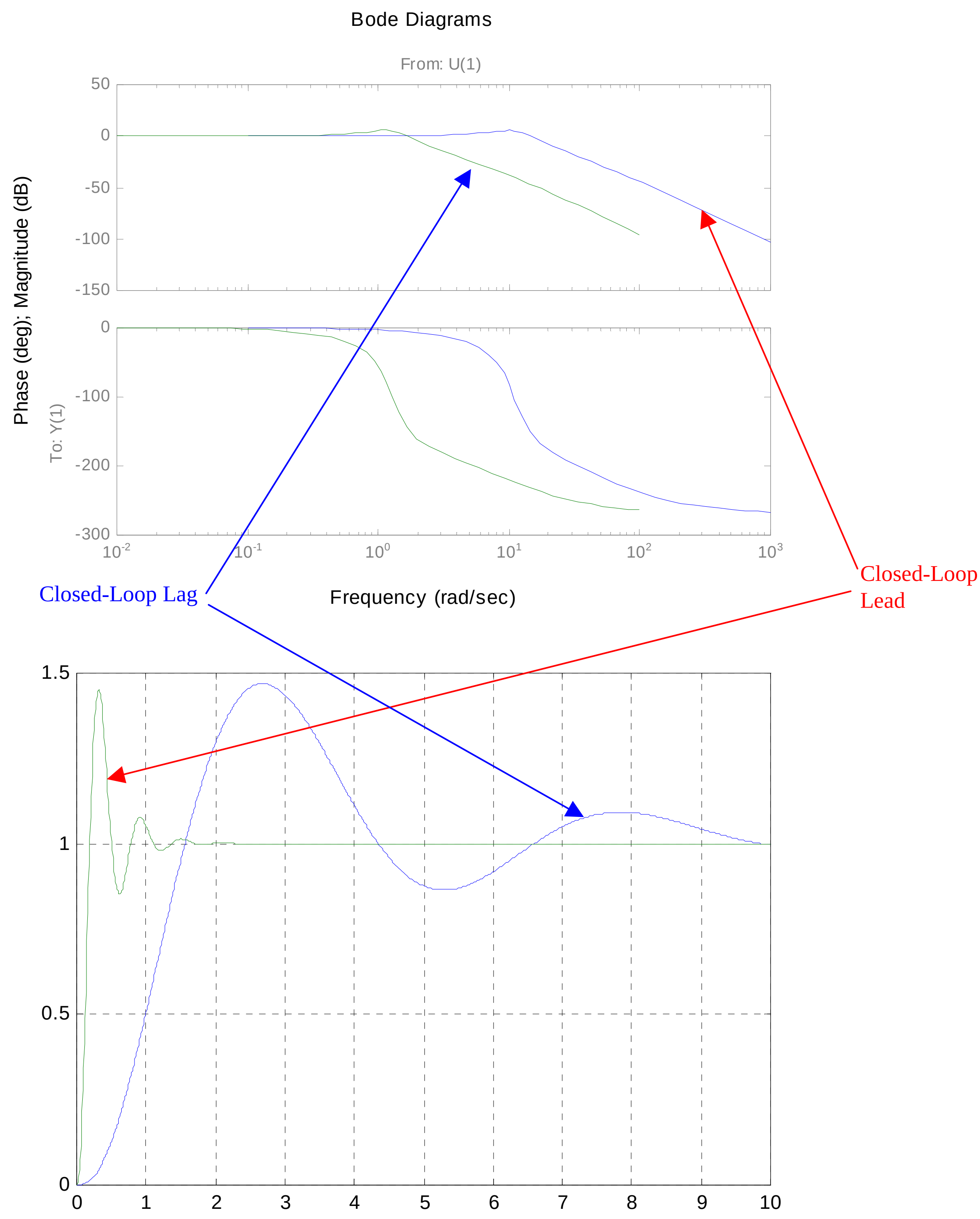
4.7040

We see that at 4.704 rad/sec, the system has sufficient phase to meet the p.m. specs, Thus, we should set the new gain cross-over frequency ω_{cgnew} = 1.132. For the lag compensator design, we will obtain an attenuation of $1/\beta$ at high frequencies. Thus, we should set β equal to the gain of $K_cG(j\ \omega_{cgnew})$ because this will make the overall gain at that frequency β/β = 1 or 0 dB and will become our new gain crossover frequency. Hence, β = 23.2 dB or 14.53. Finally, we must find T by setting the new gain crossover frequency to be a decade above our lag compensator zero. In other words, ω_{cgnew} = 10/T = 1.132 or T = 8.8343. The final lag compensator design is then given by:
$G_{lag}(s) = K_c(s)(Ts+1)/(\beta Ts+1)$ = 0.25[(8.343s+1)/(128.35s+1)]. As a final check, consider the Bode Plot of $G_{lag}G(s)$:



As one can see, the phase margin is nearly 30° at a new gain cross-over frequency of ω_{cgnew} = 1.136 rad/sec. We can also see the prominent phase dip between 0.01 and 0.1 rad/sec. Fortunately, we designed our lag compensator such that the new gain crossover frequency is well to the right of this negative or lag phase.

Just for fun, I have included the closed-loop Bode plot and step response for both the lead and lag designs below. As we will see in class, the bandwidth for the lead design is much greater than that of the lag design and the settling time of the lead design is much faster than the lag design. Thus, the lead design has better transient properties but the lag design is much more susceptible to noise.



2. Consider the lead compensator when K_c=1 (i.e., $G_c(s)$ = (Ts+1)/(α Ts+1)).

b) Find the frequency, ω_m, at which this maximum occurs (i.e, find where $\angle G_c(j\omega_m) = \phi_m$)

Solution: Note that the angle is described by $\angle G_c(j\omega) = \angle Tj\omega + 1 - \angle \alpha Tj\omega + 1 = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)$. To find frequency where maximum occurs, take the derivative of the above expression and set equal to zero (recall that the derivative of the $\tan^{-1}(x)$function is $dx/(1+x^2)$). Thus, taking the derivative we find $d\angle G_c(j\omega)/d\omega = T/(1+T^2\omega^2) - \alpha T/(1+\alpha^2 T^2\omega^2) = 0$. Thus, $T(1+\alpha^2 T^2\omega^2) - \alpha T(1+T^2\omega^2) = 0$ or $T(1-\alpha) + T(\alpha^2 - \alpha)T^2\omega^2 = 0 \Rightarrow \omega^2 = 1/\alpha T^2$. Hence, $\omega_m = 1/\sqrt{\alpha}T$

a) If ϕ_m is the maximum lead angle the compensator can supply, <u>derive</u> an expression for ϕ_m in terms of α

<u>Solution:</u> $\angle G_c(j\omega_m) = \phi_m = \angle \frac{(\frac{j}{\sqrt{\alpha}}+1)}{(j\sqrt{\alpha}+1)} = \angle \frac{2+j(\frac{1-\alpha}{\sqrt{\alpha}})}{1+\alpha}$. Hence, the sine of ϕ_m is given by

$$\frac{\mathrm{Im}(G_c(j\omega_m))}{\sqrt{\mathrm{Re}(G_c(j\omega_m)^2 + \mathrm{Im}(G_c(j\omega_m))^2}}$$

which evaluates to $\sin(\phi_m) = \frac{(\frac{1-\alpha}{\sqrt{\alpha}})}{\sqrt{2^2+(\frac{1-\alpha}{\sqrt{\alpha}})^2}} = \frac{(\frac{1-\alpha}{\sqrt{\alpha}})}{\frac{1}{\sqrt{\alpha}}\sqrt{4\alpha+1-2\alpha+\alpha^2}} = \frac{1-\alpha}{\sqrt{(1+\alpha)^2}} = \frac{1-\alpha}{1+\alpha}$

c) Find the additional gain produced by the compensator at this frequency (i.e., find $|G_c(j\omega_m)|$)

<u>Solution:</u> $|G_c(j\omega_m)| = \left|\frac{2+j(\frac{1-\alpha}{\sqrt{\alpha}})}{1+\alpha}\right| = \frac{\frac{1}{\sqrt{\alpha}}\sqrt{2^2\alpha+(1-\alpha)^2}}{1+\alpha} = \frac{\frac{1}{\sqrt{\alpha}}\sqrt{(1+\alpha)^2}}{1+\alpha} = \frac{1}{\sqrt{\alpha}}$